Using Mean-Reverting Prices and Real Options to Analyze District Heating and Combined Heat and Power in a Northern Minnesota City

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Dedication

This thesis is about District Heating in a Minnesota town. More than once during my time in Minnesota I have reflected upon the first settlers who arrived here, and wondered how they managed without having the modern conveniences (particularly heat sources) that we rely upon today.

So this thesis is dedicated to my own ancestors, Friedrich and Louisa Oberkircher, who established the farm where I grew up, and the thousands of other pioneers who settled in this country. By their tenacity and hard work they built reserves of capital from which their descendants continue to prosper.

Abstract

This paper examines alternative investment strategies for a biomass-powered District Heating (DH) system for a small city in Northern Minnesota, including Combined Heat & Power (CHP) as a method of producing both heat and electricity. Stochastic, mean-reverting commodity prices and Real Options analysis techniques are also incorporated into a financial examination of the project. The analysis finds that, given certain tax incentives, a DH+CHP project could be attractive for a profit-seeking investor. The analysis also reveals that upfront investment risk could be lowered by using an incremental Real Options approach.

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1. INTRODUCTION AND BACKGROUND

In recent decades, improvements in heat recovery and distribution technologies have transformed our ability to use low-grade heat, either directly as thermal energy or indirectly by using it to generate electricity. These developments have expanded the applications for thermal energy in two related areas: *Combined Heat and Power* production (CHP, also known as Cogeneration) and *District Energy* (DE). *CHP* is defined as the generation of useful heat and electricity from a single fuel source (usually at a location close to the end-user of the thermal energy). CHP can raise the overall fuel-to-power energy conversion efficiency from around 35% (for a conventional electricity-only generator) to above 90% (for production of electricity and heat together) (Kerr "Cogeneration.." 13). DE is a system where a thermal fluid is piped from a central location to satellite users. The satellite user could be a single firm using the fluid for industrial purposes or it could be a community of homeowners. (The fluid is most often used for heating purposes, but it may also be a coolant. DE is therefore sometimes referred to as District Heating and Cooling.)

The structure of the electricity production network in the United States, however, is not always conducive to CHP. Heat is difficult and expensive to transport over long distances. CHP is practical wherever there exists a large-enough thermal heat load in close-enough proximity to the power plant to keep transport costs and heat losses to a minimum. Over the last century electrical generation in the U.S. has tended to be concentrated at large facilities that optimize electrical efficiency, while sacrificing “residual” thermal energy. Because of their size and their impact on air quality and aesthetics, these facilities are frequently located away from population centers and other demanders of thermal energy. They typically radiate this heat (sometimes at a cost to the environment) into the air or nearby waterways. As a result, only about 8% of the 1,105 GW of nameplate electrical generation capacity installed in the United States comes from CHP. The U.S. Department of Energy estimates that an additional 110-150 GW of potential CHP capacity currently exists in the commercial and industrial sectors (Kerr “CHP/DHC” 12).

Most electricity generators in the U.S. were put into service prior to the advent of improved heat recovery technology (see Figure 1) (U.S. Energy Information Administration, “Form EIA-860...”). In fact, the oldest generator still in service dates all

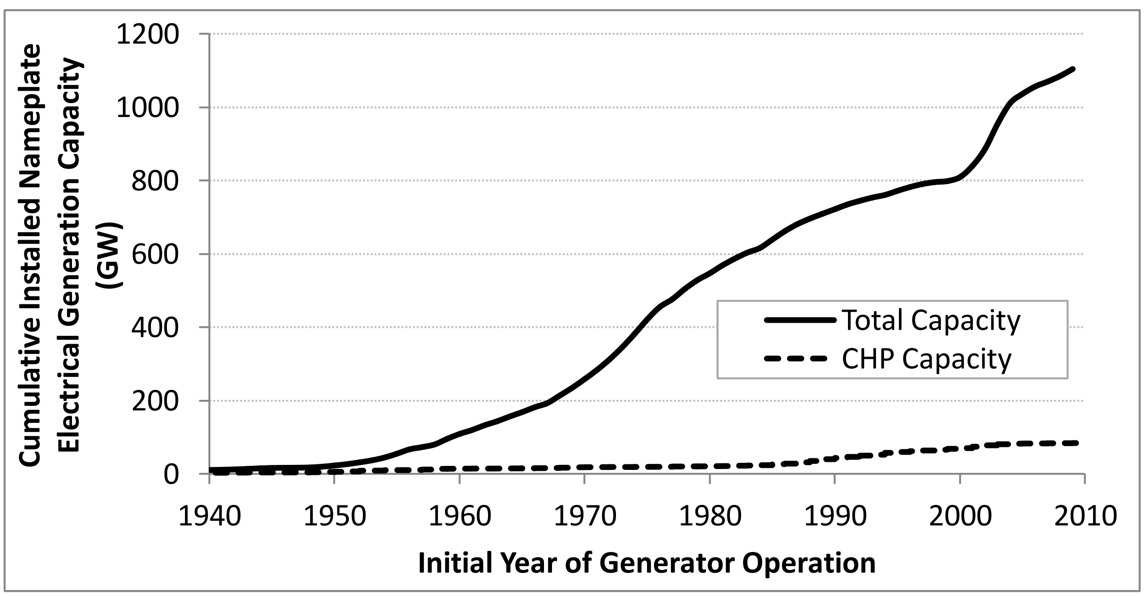

Figure 1: Total Capacity of Electrical Generation in the US over Time, with CHP broken out

the way back to 1891. So it is reasonable to assume that at the time of many generators' installation, the technology necessary to harness marginal units of thermal energy may have cost more than the energy was worth. But there are additional reasons why CHP has not taken hold in the US market. Sometimes myopia can exist that impedes industries from recognizing the value of ancillary resources. Policy regulations can also create barriers to entry in some markets. (This is particularly true in the traditionally heavily-regulated electricity market.)

A. Potential Opportunities and Challenges

However, a new model for electricity production, called Distributed Generation, could make it possible to take advantage of CHP's higher conversion efficiencies. *Distributed Generation* (DG) is a paradigm whereby electricity is produced with smaller generation units that are sized for and located closer to thermal loads (Hedman & Kaarsberg, 2001). The concepts of CHP and DE are naturally related to DG, because CHP and DE units would tend to be smaller to keep the transport distances short. Furthermore, DG combined with CHP and DE offers the opportunity to exploit local "green" energy sources that are only regionally prevalent (such as woody biomass, manure or geothermal energy) and would typically not be viable energy sources for large utilities that require concentrated amounts of easily-transportable fuel.

For a firm that is investing in an energy project, however, CHP and DE add complexity and financial risk to the investment process. We created Table 1, below, to outline some of the potential positive and negative financial impacts of investing in smaller-scale CHP and DE compared to investing in a conventional electricity-only project.

Financial Impact (+/- Compared to Electricity Generation Only)		
Input Fuel:	Local Resource/Renewable Fuel Option	(+) May reduce input price volatility (e.g.: geothermal, anaerobic digestion) (-) Potentially higher per-kW startup costs
Output Product:	Electricity	(+/-) Selling either to the real-time market or with a Power Purchase Agreement (+) If using a renewable resource, tax incentives are available for "Green Electricity" production (-) The market for "Green Electricity" subsidies and avoided CO2 emissions is variable and uncertain
	Thermal Energy	(+) Improved fuel-to-energy conversion efficiency (raised from ≈35% to ≈90%) (+/-) Output diversification may reduce revenue volatility (-) Additional investment in heat capture technology necessary (-) Necessary infrastructure for heat distribution rarely exists (-) There are not currently additional tax incentives for "Green Heat" or heat capture
Location:	Constraints	(+/-) Must be located 'near' a thermal load

Table 1: Advantages (+) and Disadvantages (-) of CHP and or DE

Despite the constraints and financial risks involved in a smaller-scale DE project, the potential to improve energy conversion efficiency by 55% or more is nontrivial. One advantage in the decision-making process is that an investor in DE may not need to decide upfront whether to invest in CHP immediately. If the investor constructs the initial DE system with the CHP option in mind, it could be added on later. The ability to separate the investment decision into periods offers flexibility for the investor. This flexibility can be analyzed as a "Real Option."

B. Analytical Tools

We will use Real Options analysis techniques to analyze the value of installing a district heating system at a location where CHP may be added at some point in the future (that is, the option exists to upgrade from the production of heat energy solely to the production of both heat and electricity). This gives the investor a chance to deal with uncertainty in the electricity market by waiting to acquire additional information about how it will develop in the future. It should be noted, however, that depending on local demand, the decision could be reversed. If electricity were the more immediately desired commodity, then the investor may decide to install an electricity generator first, and retain the option of upgrading to CHP later, after uncertainty on the thermal energy market is resolved. We, however, will adopt the former approach because we are examining a small city in northern Minnesota where high heating costs are the primary driver.

In the field of finance, a stock option is the right, but not the obligation, to buy or sell a stock at a specified price on or before some date. Specifically, a call option is the right to buy a stock in the future at the specified price. An irreversible investment that has uncertain cash flows and contains some flexibility with regard to timing represents a similar option for a firm. In the past, financial analysts relied primarily on the Net Present Value (NPV) of an investment's cash flows to judge the investment's value. But in their seminal work *Investment Under Uncertainty*, Dixit and Pindyck observed that there are many situations in which the NPV does not provide the best estimate of a project's value. Instead, they adapted the methods of valuing stock options to examine the value of delaying an investment until market uncertainty is resolved. For instance, when there is uncertainty in future cash flows, and there is flexibility to adopt a "wait and see" posture, it can make sense to ensure that today's decisions keep future options open (sometimes even if there is a cost associated with maintaining those options).

In the example of the energy project, this means that it may make sense to invest in District Heating (DH, a subset of DE) because this would be adaptable to CHP in the future, even if this upgrade is an uncertain future investment. We will explore four alternative approaches that an investor would have in order to invest in a DH/CHP project. These approaches are shown in Figure 2 (we will periodically refer back to this figure later). In each case, the investor has to deal with *two* levels of uncertainty. One level has to do with the market for heat (a factor in both the initial DH investment as well as the CHP upgrade) and the other level of uncertainty relates to the market for electricity (a factor in the CHP upgrade only).[1]

C. Objective

The objective of this paper is to examine the investment decision for a local DH/CHP project in Ely, a small city in northern Minnesota from the point of view of a hypothetical investor. We will incorporate several advanced techniques in the analysis of the investment decision. We will develop a progression for annual prices in the markets for heat and electricity (based on historical data), and will extend this progression 20 years into the future. This dual-factor progression will be stochastic, so we will define a

[1] The market for heat is the market for whatever forms of heat being displaced by the District Heating system.

probability distribution around the expected prices in each market in each year. We will further argue that these prices do not have homoskedastic distributions through time, but that they revert to a predictable mean. For instance, this could happen because consumers substitute away from energy sources that become too expensive (and the converse when their prices are low), or because high prices encourage the development of alternatives, which reduces demand for the expensive energy source. We will analyze this mean price based on historical data, and will factor it into our analysis. Finally, we will link these price progressions to a 20-year capital budgeting model for a district heating system and CHP upgrade, and we will use Real Options analysis techniques to place a value the option to expand a district heating system into a CHP electricity-producing system.

We discover that, given certain tax incentives, a DH+CHP project could be attractive for a profit-seeking investor. However, “policy risk” exists in the form of changes to government tax incentives. The stochastic price analysis also reveals significant downside risk from possible future commodity prices. The Real Options analysis shows that upfront investment risk could be lowered by using an incremental approach, but that prices need to be within favorable ranges, which we delineate as part of our sensitivity testing.

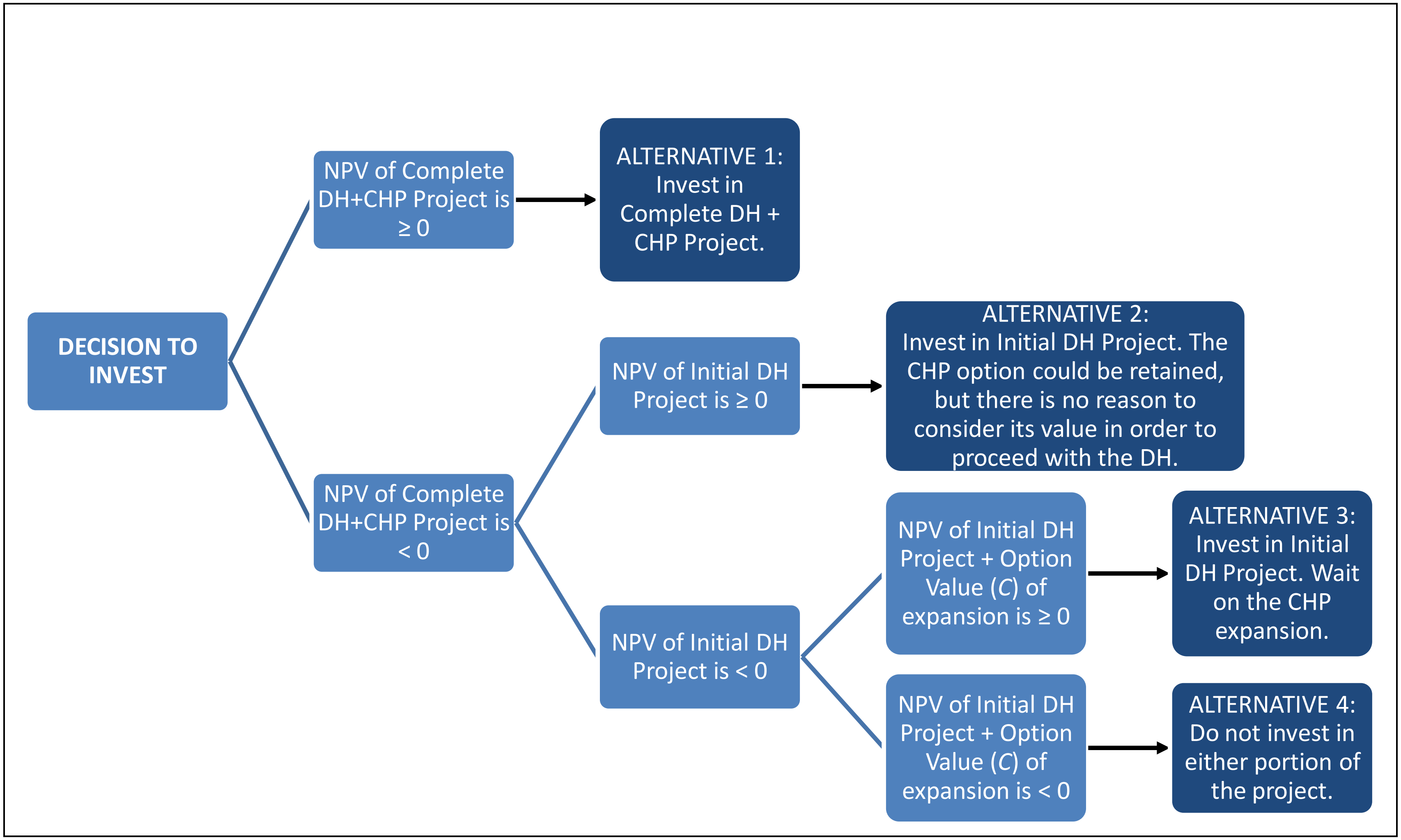

Figure 2: The investment decision process

2. METHODS

A. The Single-Factor Homoskedastic Stochastic Diffusion

Probability trees are familiar and tractable structures for evaluating stochastic price diffusions. An example of such a probability tree is shown in Figure 3 (using arbitrarily-chosen starting prices and probabilities). The investment horizon is divided into a number of periods. When the value of the single factor (*S*) is stochastic with a homoskedastic variance, from one period to the next it can either move up (by constant amounts) with a probability of p or down with a probability of $1\text{-}p=q$ (it is binomial). At $t=0$ the price is known with certainty. At $t=1$, there are two possible values, and by $t=2$ these expand to $2^t = 4$ possible values. In actuality, the homoskedastic probability "tree" in Figure 3 can be simplified into a "*lattice*" (Figure 4), because in this version an up-down motion in prices is the same as a down-up motion. So at $t=2$ there are actually three unique investment values, and so on until $t=n$, where the number of investment values = $n+1$. In a probability *lattice*, values in the same time period that are the same become joined, regardless of the series of up-or-down motions that is used to arrive at them (whereas in a probability tree they would be shown separately).

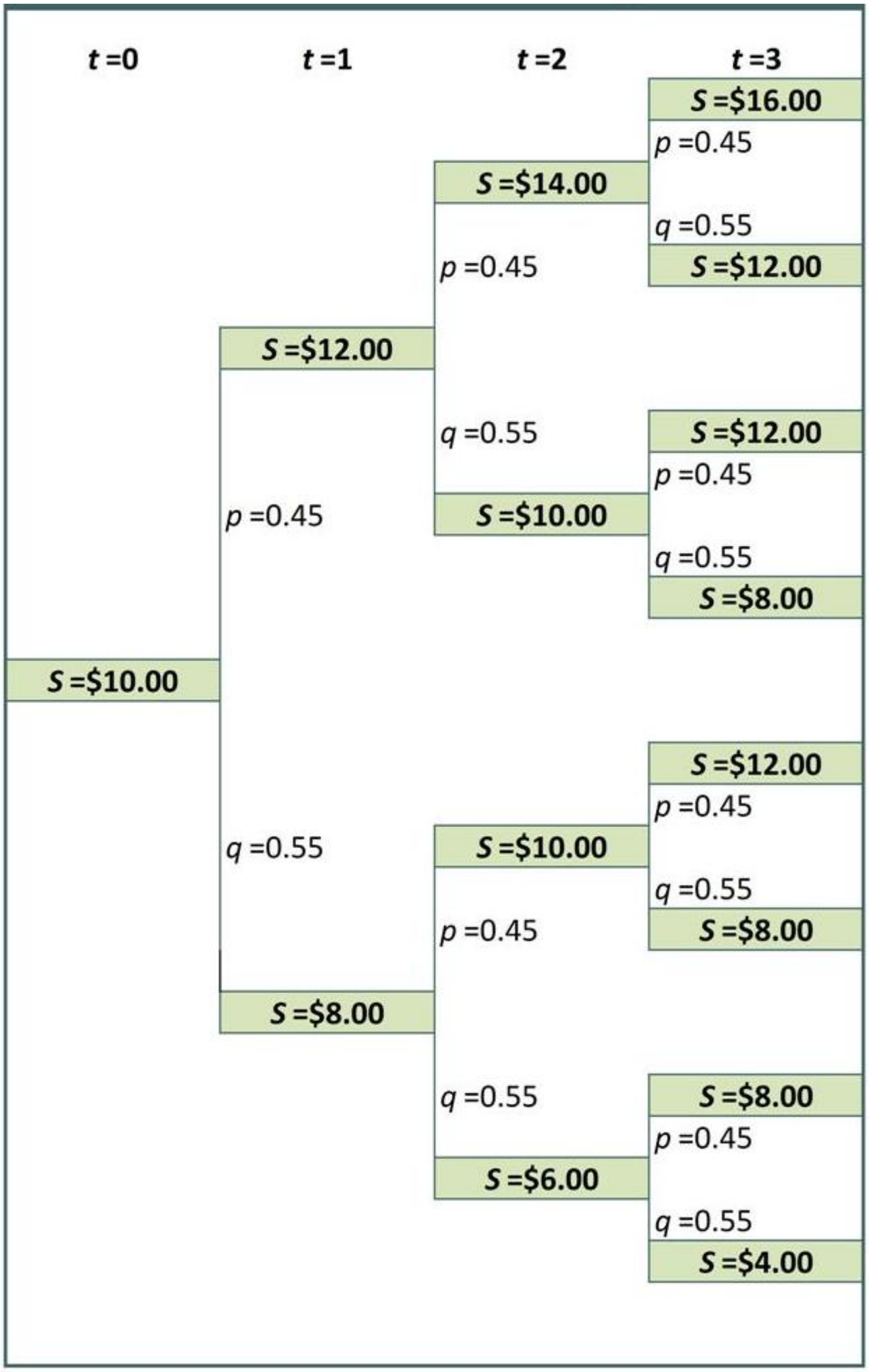

Figure 3: Example of a Binomial Probability Tree

The simplest version of the continuous-time model for this price progression is given by the equation for *Brownian motion with drift*:

$$\mathrm{d}S = \mu \mathrm{d}t + \sigma \mathrm{d}z \qquad [1]$$

where S is the value of the investment, μ is a drift parameter, and σ is a variance parameter. Commodity and stock prices are commonly assumed to be lognormally distributed, because they never fall below zero. (See Appendix A for a demonstration of the validity of this assumption.)

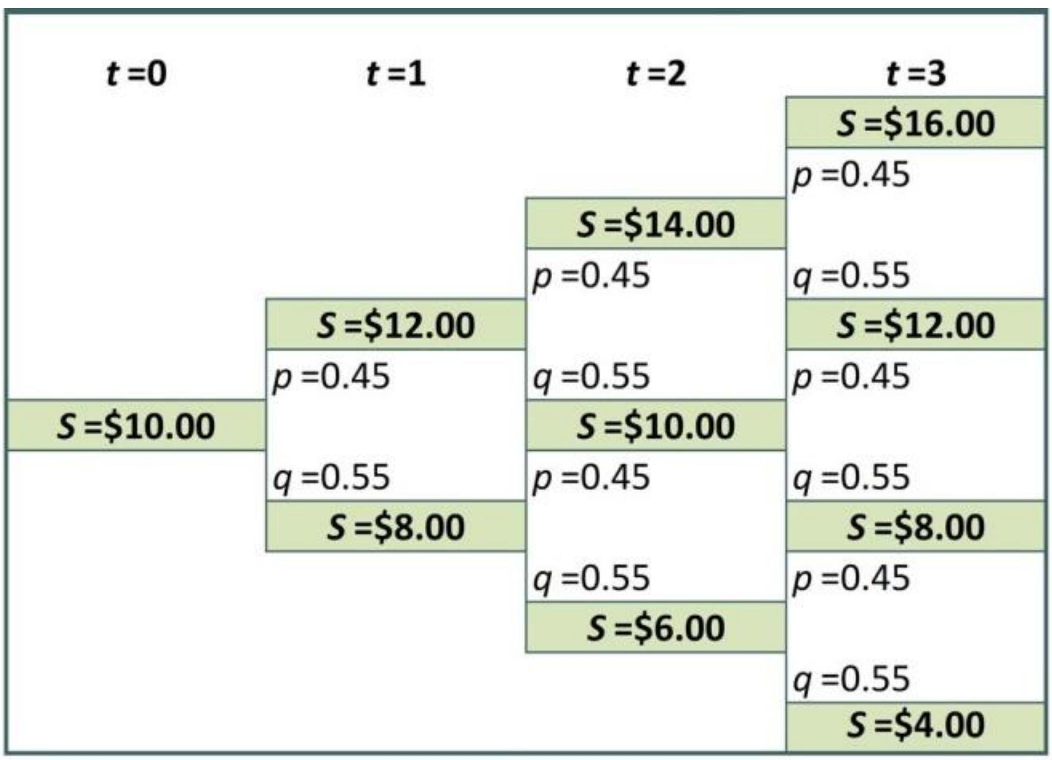

Figure 4: Example of a Binomial Probability Lattice

Using Ito's Lemma, it can be shown that the log of the investment value, S, follows the progression

$$d \ln S = \left(\mu - \frac{\sigma^2}{2}\right) \mathrm{d}t + \sigma \mathrm{d}z$$

again with μ as the drift parameter, σ as the variance (volatility) parameter, and $\mathrm{d}z$ an increment of the Wiener process with a mean equal to zero and a variance equal to $\mathrm{d}t$ (Dixit 81).

The movements up and down in this equation are given by $u = \mathrm{e}^{\sigma\sqrt{\Delta t}}$ and $d = \frac{1}{u} = \mathrm{e}^{-\sigma\sqrt{\Delta t}}$.

The probability of an up-motion is $p_u = p = \frac{u-1-r_f}{u-d}$ where r_f is the risk-free interest rate.

The probability of a down-motion is $p_d = q = 1 - p = \frac{1+r_f-d}{u-d}$.

An important observation is that because the price diffusion in this model has a constant variance, the variables p_u and p_d are the same at all nodes. So as $\Delta t \to 0$, the expected value of the investment in period t and state j can be calculated using the equation $S_{tj} = S_0 u^{t-j} d^j$.

B. The Single-Factor Heteroskedastic Stochastic Diffusion

There is theoretical and empirical evidence, however, that the diffusions of commodity prices are not homoskedastic. Rather, they tend to revert to a mean price (inflation-adjusted). Later, we solve for these mean reversion parameters empirically. The intuitive rationale is that higher commodity prices lead to lower demand and a greater incentive to increase supply by tapping unused or previously-unprofitable reserves, or by developing alternative products – all of which would bring commodity prices back down. From the opposite perspective, low commodity prices stimulate demand, which would pull prices back up. Dixit & Pindyck note that both crude oil and copper prices exhibit (albeit slow) mean reversion tendencies when one looks at their prices over 100 years or so (Dixit 77-79). As such, it would not be appropriate to model the progression of these prices (or the value of an investment that is dependent upon them) using a homoskedastic diffusion.

Instead, the progression is given by the stochastic differential equation form

$$\mathrm{d}S_t = \mu(S,t)\mathrm{d}t + \sigma(S,t)\mathrm{d}z \qquad [2]$$

where the jump size = $\mathrm{d}S_t$, the growth rate (the drift) = $\mu(S,t)$ and the standard deviation of returns (the volatility) = $\sigma(S,t)$. Again, we again use the log of S for the same reasons as before. Nelson and Ramaswamy show that for well-behaved equations where the jump sizes, drifts and variances (σ^2) converge to zero as $\Delta t \to 0$, the binomial sequence can be approximated by n periods of length Δt, and T is the time horizon so that $T = n\Delta t$.

Equation [1] can be adapted specifically to a mean-reverting process (an Ornstein-Uhlenbeck process) as follows:

$$\mathrm{d}S_t = \kappa(\bar{S} - S_t)\mathrm{d}t + \sigma \mathrm{d}z_t \qquad [3]$$

where S_t is the commodity price, κ is the mean-reversion coefficient (the speed of the reversion), $\bar{S}$ is the long-term mean price, σ is the process volatility, and $\mathrm{d}z_t$ is an increment of the Wiener process with a mean equal to zero and a variance equal to $\mathrm{d}t$ at time t. In an upcoming section we will demonstrate how to use a linear regression of historical data to solve for κ and $\bar{S}$. Again, we use the logs of S_t and $\bar{S}$, so that:

$S_t^+ \equiv S + \sqrt{\Delta t}\sigma(S,t)$ (up-move)

$S_t^- \equiv S - \sqrt{\Delta t}\sigma(S,t)$ (down-move)

$p_u \equiv \frac{1}{2} + \frac{\mu(S,t)}{2\sigma(S,t)}$ (probability of an up-move)

$1 - p_u$ (probability of a down-move)

Hahn and Dyer show that by substituting $v(S,t) = \kappa(\bar{S} - S_t)\mathrm{d}t - \frac{1}{2}\sigma^2$ for $\mu(S, t)$ and σ for $\sigma(S, t)$ in this binomial sequence, it yields the following binomial model for a *mean-reverting* process:

$S_t^+ \equiv S + \sqrt{\Delta t}\sigma$ (up-move)

$S_t^- \equiv S - \sqrt{\Delta t}\sigma$ (down-move)

$$p_u \equiv \begin{cases} \frac{1}{2} + \sqrt{\Delta t}\frac{v(S,t)}{2\sigma} & \text{if } 0 \leq \frac{1}{2} + \sqrt{\Delta t}\frac{v(S,t)}{2\sigma} \leq 1 \\ 0 & \text{if } \frac{1}{2} + \sqrt{\Delta t}\frac{v(S,t)}{2\sigma} \leq 0 \\ 1 & \text{if } 1 \leq \frac{1}{2} + \sqrt{\Delta t}\frac{v(S,t)}{2\sigma} \end{cases}$$

(probability of an up-move)

$1 - p_u$ (probability of a down-move)

Here p_u is censored as necessary to be between 0 and 1. The censoring step can be rewritten as

$$p_u = \max\left(0, \min\left(1, \left(\frac{1}{2} + \sqrt{\Delta t}\frac{v(Y,t)}{2\sigma}\right)\right)\right)$$

It should be noticed that this method of generating the heteroskedastic stochastic diffusion does not impact the possible levels of stock prices in any time, *t*, as illustrated in the probability lattice shown in Figure 4. Instead, it alters the *probabilities* of an up-motion or a down-motion at each point in time. Figure 5 depicts a hypothetical series of prices similar to Figure 4, but using the logged version of the prices with a volatility of 25% and reverting to a price of $12.00.

Depending on the time increment used, this discrete interval method of generating mean-reverting stock prices and movement probabilities will result in slightly biased estimates (biased either up or down, depending on whether the current stock price is above or below its mean-reverting value). But this bias tends toward zero by using

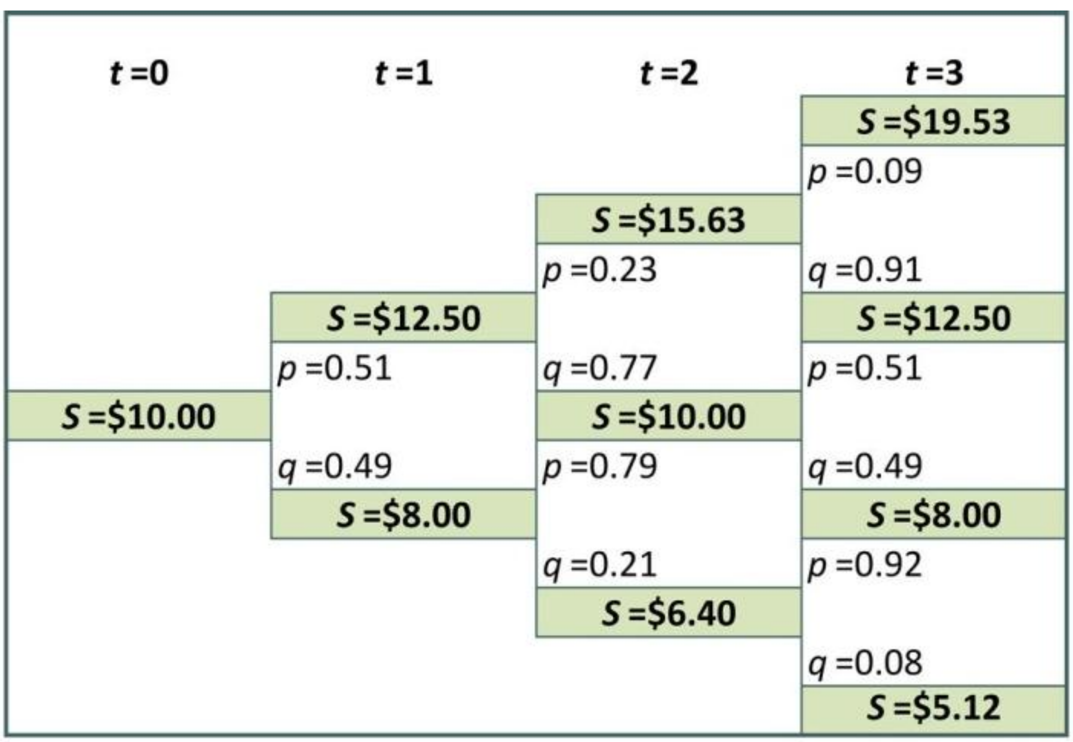

Figure 5: Example of a Binomial Logged Mean-Reverting Probability Lattice

smaller time increments. (It should be noted, however, that by using smaller time increments the number of nodes on the diffusion tree also increases geometrically.)

C. The Dual-Factor Heteroskedastic Stochastic Diffusion

When considering a real-world investment, however, the decision is often between two production alternatives that are dependent upon different stochastic (and mean-reverting) price diffusions. We will call these Input X and Input Y. In this case, the price diffusions for the individual inputs, along with the probabilities of their upward & downward motions must be combined to determine how they affect the profitability of the overall investment. In this case, though, we must move beyond the single factor binomial model, because in each time period the motion of the investment's value can be influenced in *four* different directions. Either:

P_X can go up and P_Y can go up (*uu*),

P_X can go up and P_Y can go down (*du*),

P_X can go down and P_Y can go up (*ud*), or

P_X can go down and P_Y can go down (*dd*).

The individual binomial diffusions for the prices of Input X and Input Y must therefore be combined into a *quadrinomial* diffusion, as depicted in Figure 6 (again, Figure 6 uses arbitrarily-chosen starting prices and probabilities purely for example purposes.)

In a situation where the prices of X and Y are mean-reverting, they each follow a lognormal Ornstein-Uhlenbeck process, i.e.:

$\mathrm{d}X_t = \kappa_X(\bar{X} - X_t)\mathrm{d}t + \sigma_X \mathrm{d}z_{Xt}$ and

$\mathrm{d}Y_t = \kappa_Y(\bar{Y} - Y_t)\mathrm{d}t + \sigma_Y \mathrm{d}z_{Yt}$

Of course it is also important to know the correlation between the prices of X_t and Y_t, which is given by

$$\mathrm{d}z_X \mathrm{d}z_Y = \rho_{XY} \mathrm{d}t.$$

Following the derivation in Hahn-Dyer's Appendix A (2008), the increments (Δ_X and Δ_Y) and joint probabilities on each branch of the quadranomial tree are:

$$\Delta_X = \sigma_X \sqrt{\Delta t} \quad [4]$$

$$\Delta_Y = \sigma_Y \sqrt{\Delta t} \quad [5]$$

$$p_{uu} = \frac{1}{4} \cdot \frac{\Delta_X \Delta_Y + \Delta_Y v_X \Delta t + \Delta_X v_Y \Delta t + \rho \sigma_X \sigma_Y \Delta t}{\Delta_X \Delta_Y} \quad [6]$$

$$p_{ud} = \frac{1}{4} \cdot \frac{\Delta_X \Delta_Y + \Delta_Y v_X \Delta t - \Delta_X v_Y \Delta t - \rho \sigma_X \sigma_Y \Delta t}{\Delta_X \Delta_Y} \quad [7]$$

$$p_{du} = \frac{1}{4} \cdot \frac{\Delta_X \Delta_Y - \Delta_Y v_X \Delta t + \Delta_X v_Y \Delta t - \rho \sigma_X \sigma_Y \Delta t}{\Delta_X \Delta_Y} \quad [8]$$

$$p_{dd} = \frac{1}{4} \cdot \frac{\Delta_X \Delta_Y - \Delta_Y v_Y \Delta t + \Delta_X v_Y \Delta t + \rho \sigma_X \sigma_Y \Delta t}{\Delta_X \Delta_Y} = 1 - p_{uu} - p_{du} - p_{ud} \quad [9]$$

where v_X and v_Y are the drift factors equal to:

$$v_X(X,t) = \kappa_X(\bar{X} - X_t) - \frac{1}{2}\sigma_X^2 \text{ and} \quad [10]$$

$$v_Y(Y,t) = \kappa_Y(\bar{Y} - Y_t) - \frac{1}{2}\sigma_Y^2 \quad [11]$$

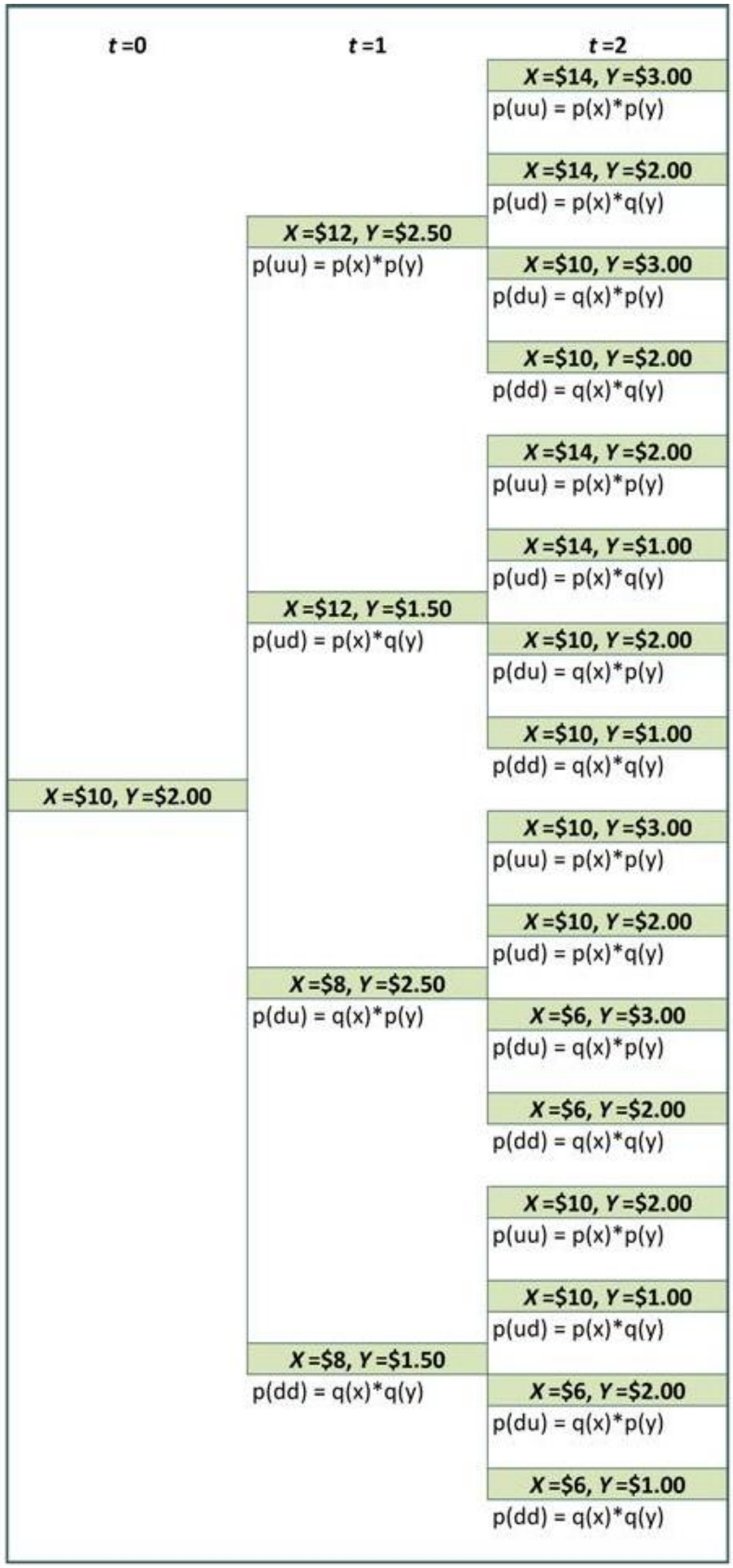

Figure 6: Example of a Quadrinomial Probability Tree

Just as in the single-factor heteroskedastic diffusion where p_u needed to be censored to be between 0 and 1, these probabilities must also be censored. The problem, however, is that these are the joint probabilities for an *uu*, *ud*, *du*, or *dd* motion. We should not censor the joint probabilities, but rather the conditional probabilities of the motion in the prices of X and Y independently, i.e.: For X we should censor: p_u and p_d. For Y we should censor $p_{u|u}$, $p_{d|u}$, $p_{u|d}$ and $p_{d|d}$. So we must first decompose the joint probabilities into their conditional parts. This is accomplished using Bayes' Rule:

$$p(X_{t+1} \cap Y_{t+1}) = p(Y_{t+1}|X_{t+1})p(X_{t+1})$$

We already know the marginal probabilities for Input X. They are:

$$p_u = \frac{1}{2} + \frac{v_X \Delta t}{\Delta_X} \text{ and} \qquad [12]$$

$$p_d = \frac{1}{2} - \frac{v_X \Delta t}{\Delta_X} \qquad [13]$$

we need only divide Equations [5]-[8] by these marginal probabilities to yield:

$$p_{u|u} = \frac{1}{2} \cdot \frac{\Delta_X(\Delta_Y + \Delta t v_Y) + \Delta t(\Delta_Y v_X + \rho\sigma_X\sigma_Y)}{\Delta_Y(\Delta_X + \Delta t v_X)} \qquad [14]$$

$$p_{d|u} = \frac{1}{2} \cdot \frac{\Delta_X(\Delta_Y - \Delta t v_Y) + \Delta t(\Delta_Y v_X - \rho\sigma_X\sigma_Y)}{\Delta_Y(\Delta_X + \Delta t v_X)} \qquad [15]$$

$$p_{u|d} = \frac{1}{2} \cdot \frac{\Delta_X(\Delta_Y + \Delta t v_Y) - \Delta t(\rho\sigma_X\sigma_Y + \Delta_Y v_X)}{\Delta_Y(\Delta_X - \Delta t v_X)} \qquad [16]^2$$

$$p_{d|d} = \frac{1}{2} \cdot \frac{\Delta_X(\Delta_Y - \Delta t v_Y) + \Delta t(\rho\sigma_X\sigma_Y - \Delta_Y v_X)}{\Delta_Y(\Delta_X - \Delta t v_X)} = 1 - p_{u|u} - p_{u|d} - p_{d|u} \qquad [17]^5$$

The marginal probabilities represented by equations [11]-[16] can then be censored outside of 0 and 1 as appropriate. After they are censored, they can be re-multiplied to become the joint probabilities for each branch of the probability tree.

[2] Note: In Hahn & Dyer (2006) the equations for $p_{u|d}$ and $p_{d|d}$ are not accurately reported. They are corrected here.

As we have noted, in a binomial price diffusion it is common to generate a visual representation of a recombining lattice like Figure 4. With two stochastic prices and a quadrinomial diffusion, however, it would be complicated to represent the lattice visually – even though the values on some branches are the same.

The key to simplifying the diffusion calculations is to notice that (following equations [10]-[17]) the probabilities of upward and downward motions are the same *whenever both stock prices are the same*, because X_t and Y_t in equations [10] and [11] will be the same. This enables nodes to recombine and limits the number of independent calculations that need to be made. (In Figure 6, for instance, it reduces the number of nodes in t = 2 from 16 to 10.)

We have already noted that when analyzing a mean-reverting process, Δ_X and Δ_Y do not change at each node, but rather the *probabilities* of upward and downward motions at those nodes change. Using S_{X0}, S_{Y0}, Δ_X and Δ_Y, then, we can set up a matrix of possible combinations of stock values, and the probabilities of upward or downward motions at those nodes (see Figure 7). Figure 7 shows the probabilities of *uu*, *ud*, *du* and *dd* price motions at each price combination. It is a lognormal quadrinomial progression for three future periods of residential electricity prices and residential heating oil prices (per mmBTU). Note that $\sigma = \sigma\sqrt{t}$ where t=1. One peculiarity of the table is that at each node the number of sigma deviations from S_0 for Input X and Input Y must both either be even or odd. There can be no nodes such as ($S_{X0} + \mathbf{1}\sigma$), ($S_{Y0} + \mathbf{2}\sigma$). This is because in each period the prices of both Input X and Input Y must move either up or down. The combination ($S_{X0} + \mathbf{0}\sigma$), ($S_{Y0} + \mathbf{2}\sigma$), on the other hand, is possible. From that node the

Product X = Residential Electricity (mmBTU) | Product Y = Residential Heating Oil (mmBTU)

S_{0X}=	\$ 33.76	S_{0Y}=	\$ 25.23
σ_X=	3.8%	σ_Y=	17.5%
$Sbar_X$=	\$ 32.45	$Sbar_Y$=	\$ 16.86
$kappa_X$=	0.109	$kappa_Y$=	0.073
rho_{XY}=	12.0%		

Probability of UP_X-UP_Y Motion

		\$ 14.93	\$ 17.78	\$ 21.18	\$ 25.23	\$ 30.05	\$ 35.80	\$ 42.64
		$S_{0Y} - 3\sigma$	$S_{0Y} - 2\sigma$	$S_{0Y} - \sigma$	S_{0Y}	$S_{0Y} + \sigma$	$S_{0Y} + 2\sigma$	$S_{0Y} + 3\sigma$
$S_{0X} - 3\sigma$	\$ 30.16							
$S_{0X} - 2\sigma$	\$ 31.31		27.38%		23.75%		20.12%	
$S_{0X} - \sigma$	\$ 32.51			22.84%		19.21%		
S_{0X}	\$ 33.76		21.93%		18.30%		14.67%	
$S_{0X} + \sigma$	\$ 35.05			17.38%		13.75%		
$S_{0X} + 2\sigma$	\$ 36.40		16.47%		12.84%		9.21%	
$S_{0X} + 3\sigma$	\$ 37.79							

Probability of UP_X-$DOWN_Y$ Motion

		\$ 14.93	\$ 17.78	\$ 21.18	\$ 25.23	\$ 30.05	\$ 35.80	\$ 42.64
		$S_{0Y} - 3\sigma$	$S_{0Y} - 2\sigma$	$S_{0Y} - \sigma$	S_{0Y}	$S_{0Y} + \sigma$	$S_{0Y} + 2\sigma$	$S_{0Y} + 3\sigma$
$S_{0X} - 3\sigma$	\$ 30.16							
$S_{0X} - 2\sigma$	\$ 31.31		26.85%		30.48%		34.11%	
$S_{0X} - \sigma$	\$ 32.51			25.94%		29.57%		
S_{0X}	\$ 33.76		21.40%		25.03%		28.66%	
$S_{0X} + \sigma$	\$ 35.05			20.48%		24.11%		
$S_{0X} + 2\sigma$	\$ 36.40		15.94%		19.57%		23.20%	
$S_{0X} + 3\sigma$	\$ 37.79							

Probability of $DOWN_X$-UP_Y Motion

		\$ 14.93	\$ 17.78	\$ 21.18	\$ 25.23	\$ 30.05	\$ 35.80	\$ 42.64
		$S_{0Y} - 3\sigma$	$S_{0Y} - 2\sigma$	$S_{0Y} - \sigma$	S_{0Y}	$S_{0Y} + \sigma$	$S_{0Y} + 2\sigma$	$S_{0Y} + 3\sigma$
$S_{0X} - 3\sigma$	\$ 30.16							
$S_{0X} - 2\sigma$	\$ 31.31		17.14%		13.51%		9.88%	
$S_{0X} - \sigma$	\$ 32.51			18.05%		14.42%		
S_{0X}	\$ 33.76		22.60%		18.97%		15.34%	
$S_{0X} + \sigma$	\$ 35.05			23.51%		19.88%		
$S_{0X} + 2\sigma$	\$ 36.40		28.05%		24.42%		20.79%	
$S_{0X} + 3\sigma$	\$ 37.79							

Probability of $DOWN_X$-$DOWN_Y$ Motion

		\$ 14.93	\$ 17.78	\$ 21.18	\$ 25.23	\$ 30.05	\$ 35.80	\$ 42.64
		$S_{0Y} - 3\sigma$	$S_{0Y} - 2\sigma$	$S_{0Y} - \sigma$	S_{0Y}	$S_{0Y} + \sigma$	$S_{0Y} + 2\sigma$	$S_{0Y} + 3\sigma$
$S_{0X} - 3\sigma$	\$ 30.16							
$S_{0X} - 2\sigma$	\$ 31.31		28.63%		32.26%		35.89%	
$S_{0X} - \sigma$	\$ 32.51			33.17%		36.80%		
S_{0X}	\$ 33.76		34.08%		37.71%		41.34%	
$S_{0X} + \sigma$	\$ 35.05			38.62%		42.25%		
$S_{0X} + 2\sigma$	\$ 36.40		39.53%		43.16%		46.79%	
$S_{0X} + 3\sigma$	\$ 37.79							

Figure 7: Example of a Nodal Probability Matrix for Two Stocks with Input X prices in time *t* on the Y-axis and Input Y prices in time *t* on the X-axis.

possibility of an *uu* motion is 14.67%, an *ud* motion is 28.66%, a *du* motion is 15.34% and a *dd* motion is 41.34%, for a sum of 100%.

D. Solving for Expected Prices and Distributions in Each Time Period

We now have all the information needed to solve for expected price values in each time period and we can also define the distributions around that price: we know the prices in Period 0, and we know the probability of moving one standard deviation in either direction into Period 1. This allows us to calculate the expected price and the distribution around it in Period 2, and so on until we reach the end of our time horizon. In Visual Basic we have written a program for Microsoft Excel that can run these calculations at least 50 periods into the future within normal computing times. A copy of the code can be found in Appendix B.

The Visual Basic macro works forward in time, starting in Period t and calculating the probability of advancing to each possible node in Period t +1 that is, $p(X_{t+1} \cap Y_{t+1})$, using Equations [11]-[16], and calculates the cumulative probabilty by multiplying $p(X_{t+1} \cap Y_{t+1})$ by the probability of reaching the current node, $p(X_t \cap Y_t)$. For situations where one node might be reachable by multiple "routes" within a single time period, the macro adds all like nodes together. The result is a price-and-probability lattice for each combination of commodity prices for Inputs X and Y. We can make the lattice easier to represent visually by separating Input X from Input Y – in other words, by calculating $p(X_{UP}) = p(uu) + p(ud)$ for Input X or $p(Y_{UP}) = p(uu) + p(du)$ for Input Y, etc. (see Figure 8).

Next we are able to use this information to calculate the expected Prices of Input X and Input Y in each time period, as well as the standard deviation of the distribution around that Expected Value. (Note that this calculation is done with the logs of the prices.) The formula for calculating the expected price in each time period (*t*) using each state (*j*) is:

$$\mathcal{E}(S)_t = \sum_{j=0}^{t+1} S_{tj} \cdot p_{tj}$$

The formula for calculating the standard deviation (σ) around the expected price in each time period is: $\sigma_t = \sum_{j=0}^{t+1} \left[S_{tj} - \mathcal{E}(S)_t\right]^2 \cdot p_{tj}$.

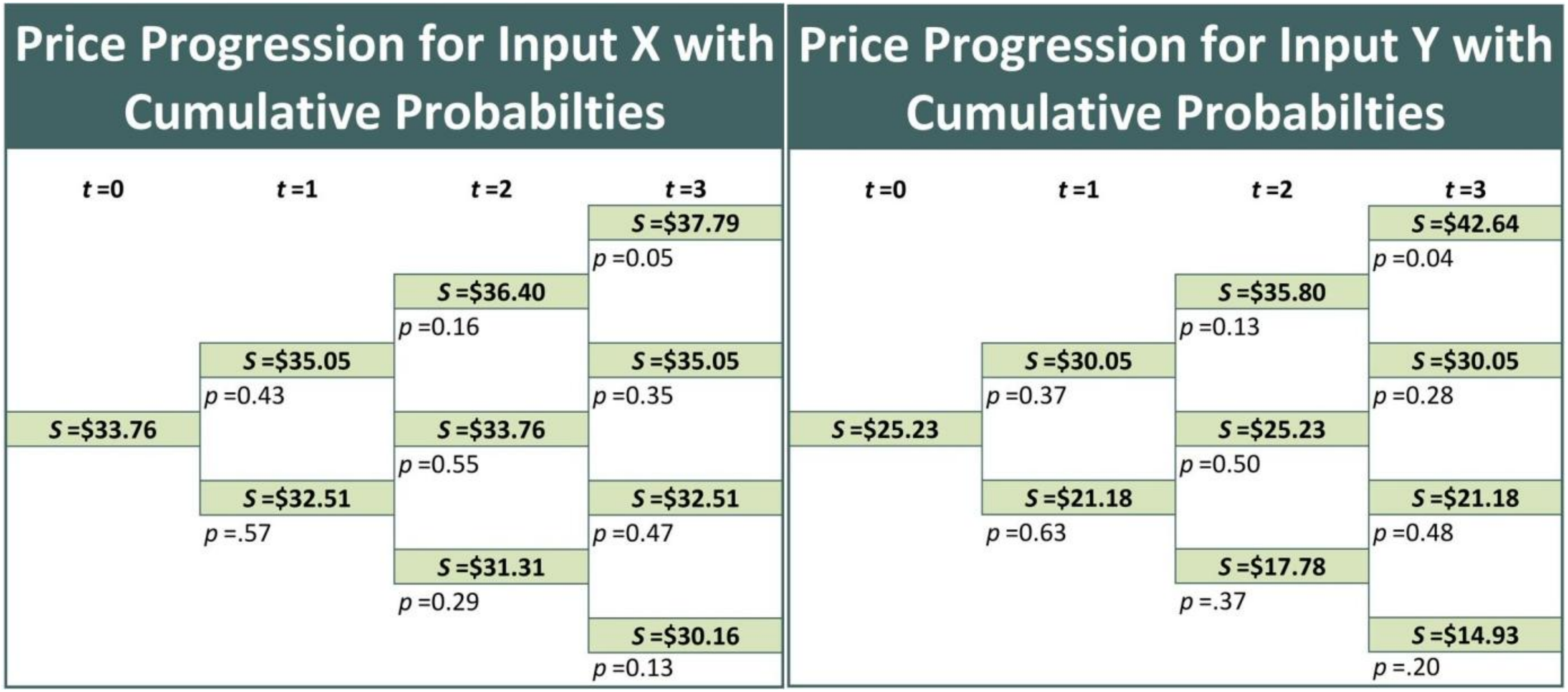

Figure 8: Example Price Progression for Two Stocks (X and Y) Shown in Separate Panels

Table 2, below, shows the expected prices and standard deviations for the distributions in this example by time period. Recall that the price of Input X is reverting to \$32.45 and Input Y to \$16.86.

	t =0	t =1	t =2	t =3
Input X	\$ 33.76	\$ 33.59	\$ 33.44	\$ 33.30
Std. Dev. (σ)	-	3.7%	5.0%	5.8%
Input Y	\$ 25.23	\$ 24.13	\$ 23.15	\$ 22.28
Std. Dev. (σ)	-	16.9%	23.1%	27.3%

Table 2: Example Expected Prices and Standard Deviations

E. Solving for Mean Reversion Parameters

Equations [9]-[16] contain certain parameters related to mean reversion that must be solved for based on historical data. The technique for solving for these parameters using discrete-time data is given by Dixit & Pindyck in their 1994 book, "Investment Under Uncertainty" (page 76). Dixit and Pindyck begin with the formula for a standard mean-reverting process, the Ornstein Uhlenbeck process.[3] The formula for the continuous-time Ornstein-Uhlenbeck Process is $dx = \eta (\bar{x} - x)\, dt + \sigma dz$, with η being the speed of the reversion and $\bar{x}$ being the mean-reverting level of x. They use this equation and its associated variance in time t to solve for the prices and variances as Δt approaches zero, and then they reduce the equations to forms that are usable from one discrete point

[3] Note that here we adopt the same equations as Dixit and Pindyck, including the lower-case x. This is merely to distinguish the x in Dixit and Pindyck's theoretical equations from the X in the examples, which refers to some specific commodity.

in time to the next. Eventually this equation becomes:

$$x_t - x_{t-1} = \bar{x}(1 - e^{-\eta}) + (e^{-\eta} - 1)x_{t-1} + \epsilon_t, \quad [18]$$

where the error term ϵ_t is normally distributed with a standard deviation of σ_ϵ, and

$$\sigma_\epsilon^2 = \frac{\sigma^2}{2\eta}(1 - e^{-2\eta}) \quad [19]$$

Equation 18 can then be evaluated using discrete-time data and an ordinary least squares regression in which the difference of the prices between two time periods is the dependent variable. Using regression results, equation 18 becomes: $X_t - X_{t-1} = a + bX_{t-1} + \epsilon_t$, and the mean reversion price, $\bar{X}$, is then given by the estimates $\frac{-\hat{a}}{\hat{b}}$. The speed of this reversion is $\hat{\eta} = -\log(1 + \hat{b})$, and the volatility around this progression is given by $\hat{\sigma} = \hat{\sigma}_\epsilon\sqrt{\frac{2\cdot\log(1+\hat{b})}{(1+\hat{b})^2-1}}$ (which is the same as Equation 19, but with $-\log(1 + \hat{b})$ substituted for η.)[4]

Since we are operating under the assumption that the changes in prices are lognormally distributed, we use the logs of X_t in these calculations, and then convert these back to the final prices by exponentiating them (again, see Appendix A).

F. Integration into the Capital Budgeting Model

After using the procedure detailed above to attain the prices of the underlying commodities in each time period, we can use them in an annualized capital budgeting

[4] Note: in Dixit and Pindyck the numeral two on the top of the fraction underneath this radical is erroneously absent, but if one makes the substitution of $-\log(1 + \hat{b})$ for η into Equation 19, and reduces it, solving for σ_ϵ rather than σ_ϵ^2, one finds that it belongs.

model to assess the installation phases of district heating and combined heat and power in Ely, Minnesota (population 3,400). In the capital budgeting model, there is some utility company or other investor that undertakes the project to install district heating pipes and a biomass-fuelled boiler (Phase 1). We then analyze the option that the utility would have to upgrade the boiler to enable them to also produce electricity. The heat from the electricity generation (which would normally be wasted into the environment) could be captured circulated through the district heating system (Phase 2).

We use a "zero economic profit" assumption to say that the value of the heat is its avoided cost to the consumer. That is, the utility company would be able to sell the heat from its district heating system at a price equal to the homeowner's or business' costs from their existing "heat portfolio." The heat portfolio is represented by the complement of heating fuels currently used in Ely (see the Application of Methods section for more detail). This assumption implies that the heat customer's changeover costs would be zero. In reality, however, a consumer who is linking into a district heating system may be required to pay the costs for piping from their home to the street and for heat exchangers in their home. This means that homeowners would only be willing to switch to district heating if the cost per-mmBtu of supplied heat were *lower* than the current avoided costs of their heat portfolio. We effectively account for this, however, by adding the homeowners' interconnection costs to the utility's capital costs. (Note that the water circulating throughout town's district heating pipes would not be not the same hot water that homeowners would have circulating inside their houses. This helps control the quality of the fluid circulating throughout the town, and reduces the likelihood of supply interruptions or liability complications if there were a leak or a blockage within the house an individual homeowner. Instead, a heat exchanger would be placed inside the home in order to keep the systems separate. The one-time capital cost of home conversion and connection to the utility's heat grid would be around \$6,255 if the home currently has hot-water heat, and around \$7,205 integrating the district heating into a home heated with forced-air. We use an average conversion cost of \$6,730 per building.) (Hartley "Building...")

3. APPLICATION OF METHODS

A. Capital Equipment and Project Phases

Estimates of capital costs and equipment sizing for the town were obtained from Charles E. Hartley of the engineering firm LHB and Associates (Hartley "2 Ely DH..."). These costs are outlined in Table 3. Mr. Hartley was contracted by a local community group in Ely to produce a detailed feasibility study of the potential for district heating and CHP. The CHP base load system would be fueled using woody biomass from the forests surrounding Ely. Much of the necessary biomass is already available as waste from the lumbering industry and from clearing projects conducted by the U.S. Forest Service as part of its Firewise programs. Currently, most of the residual top and branch biomass (TBB) is left in the forest after being processed on-site and over time is decomposed and recycled into forest soils. But forest analysis shows that using current harvest practices, more than enough biomass could be gleaned from surrounding forests without compromising the productivity and diversity of the forest's ecosystem or wildlife habitat (Domke).

Upfront Capital Costs			
	Total	**Phase 1 Heat Only** (253 Homes/Businesses)	**Phase 2 CHP** (+127 Homes/Businesses)
Wood Receiving & Storage	**$376,445**	$376,445	
Wood & Fossil Fuel Boilers	**$1,399,300**	$1,399,300	
Building	**$1,837,510**	$1,779,382	$58,127
Mechanical & Electrical	**$887,917**	$656,046	$231,871
Engineering, Civil & Owners Costs	**$1,937,893**	$1,121,283	$816,610
TO Heater	**$4,003,928**		$4,003,928
ORC	**$2,180,000**		$2,180,000
District Heat Piping	**$4,373,493**	$3,462,348	$911,144
Homeowner Conversion Costs	**$2,557,552**	$1,702,791	$854,761
Total	**$19,554,037**	$10,497,596	$9,056,441
Annual Operating Costs			
Maintenance		$40,000	+ $85,000
Labor		$40,000	+ $30,000
Insurance & Misc.		$40,000	+ $30,000
Year-1 Electricity	*can vary in model. These operating costs use a starting purchase price of $0.06/kWh for electricity, $27/ton for wood chips and $230/ton for pellets.	$62,726	total $0.00 (self-sufficient)
Year-1 Wood Fuel		$1,518,641	total $793,596

Table 3: Summary of DH and CHP costs and Phases

Phase 1 of the project would be to install the two wood pellet-fuelled boilers and two fossil fuel boilers on the district heating site, as well as the site structures and fuel unloading and storage facilities. (See Table 3 for a summary of these facilities and their costs, as compiled by Mr. Hartley. Cost and performance information is used as in Mr. Hartley's spreadsheet titled "2 Ely DH Eng Study Spreadsheet," except that we separate the equipment costs into the equipment needed by phase, and add in the homeowner conversion costs) Most of the district heating pipes would also be installed in town in the first phase. In Phase 1 the wood pellet-fueled boilers with an efficiency rating of 74%[5] would have a peaking capacity of 20.4 mmBtu/hr and would be able to cover the heating load for all of Ely's public buildings[6] as well as approximately 253 private businesses and residences, while leaving 25% capacity unused for unexpected demand fluctuations. In Phase 2 these pellet-fueled boilers would become peaking units that do not operate constantly (which is why they are pellet-fueled boilers and not wood chip fueled-boilers: pellet fuel is dryer, cleaner and more consistent than wood chips, making it possible in Phase 2 to easily power them up and down to cover peak demand). The two fossil fuel boilers would not be used regularly, and would operate only in the case of an outage with the pellet-fuelled boiler.

In Phase 2 we analyze the option that the utility would then have to upgrade the boilers in order to enable it to also produce electricity. The heat from the electricity

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[5] This is the final adjusted efficiency rating. The pellet boiler has a nameplate efficiency rating of 77%, but engineers must account for other system losses before the heat enters the grid.

[6] This includes the Vermilion Community College, the Hospital and Clinic, the School, City Hall, and the apartment complexes Pioneer, Sibley and Zenith.

generation (which would normally be wasted into the environment) could then be captured and circulated through the existing district heating system (CHP). In Phase 2, the wood pellet-fueled boilers would become the peaking units, and a CHP unit from the company Turboden that produces electricity using an organic rankine cycle (ORC) would serve most of the town's base load heating needs using waste heat from the production of 8,657 MWh of electricity per year (that is, an operating capacity of 1,043 kWe[7] and an online capacity factor of 94.7% to account for 19 cumulative days per year of potential maintenance and other downtime). The CHP unit would run on less-expensive TBB from the forests surrounding Ely. This would enable the utility to expand service to an additional 127 private businesses and residences while continuing to maintain 25% unused capacity for fluctuations and future growth. (In both phases the fossil-fuel boilers would be on-site as backups, but would not be used unless there was a complete failure of the biomass-based heating units.)

B. Cash Flow Model Explanation

The cash flow model into which all these prices feed is annualized and is used to calculate common financial metrics for the project such as Payback Periods, Net Present Values (NPV) and Modified Internal Rates of Return (MIRR).[8] It also enables sensitivity analyses and scenario analyses to explore the impacts of debt financing for the project or grant funding. The analysis is conducted in inflation-adjusted dollars,[9] meaning prices

[7] kWe = kilowatts of electricity, as opposed to kWt = thermal kW
[8] The formulas for these metrics are contained in Microsoft Office Excel 2007.
[9] Adjusted using the annual gross domestic product figures from the National Income and Product Accounts Table 1.1.9. (Implicit Price Deflators for Gross Domestic Product)

only change if there are technological changes or shifts in the supply or demand curves causing prices to change in real terms. Therefore, non-fuel factors such as labor, operating, and maintenance costs do not change over time in the model since it is not conceivable that the DH & CHP system would have a noticeable impact on the labor supply in Ely or on the market for heating equipment.

In all scenarios, the project is 100% debt-financed for the life of the project (20 years). In order to identify an appropriate interest rate on the debt, we must make assumptions about the nature of the actor in this scenario. We have chosen to assume a corporate actor, even though a municipality or cooperative could also be the actor. The main reason for choosing a corporate actor, however, is the fact that municipalities and cooperatives are exempt from federal and state income taxes (Bailey), so if we chose a municipal or cooperative actor, we would not be able to investigate the impact in the model of the electricity production tax credit (PTC) or accelerated tax depreciation. Since these are the main tools used in the United States to incentivize renewable electricity production, we thought it was important to include them.[10]

Corporate actors are able to loan investment capital at a variety of rates, depending on their credit-worthiness (or their quality). The U.S. Department of the Treasury offers a comprehensive dataset for bonds of many different maturity horizons called the HQM Corporate Bond Yield Curve. It assumes a high-quality actor in the

[10] Note that since we began this paper, Congressional authorization for the PTC lapsed. It was not renewed at the end of 2013. An extension of the PTC is part of tax bills currently being debated in Congress (American Wind Energy Association). Because of its historical importance in the United States and because of the likelihood of its return, we continue with it in the model at its 2010 rate.

range of A to AAA. Taking the most recent five years' worth of annual yield rates for bonds with a maturity horizon of 20 years, we deflated them using the formula $1 + R = (1 + r) \times (1 + h) \;\rightarrow\; r = \frac{(1+R)}{(1+h)} - 1$, where r = the real interest rate, R = the nominal bond yield rate, and h = the inflation rate adjustment (Ross, Westerfield and Jordan 220).[11] This results in a deflated annual cost of r = 4.25%, which is used as the loan rate, the discount rate in the NPV formula, the before-tax hurdle rate of return, and both the finance rate and the reinvestment rate in the MIRR formula.

The income tax rate used is the sum of the state and federal corporate income tax rates. In Minnesota, this is a flat 9.8% "corporate franchise tax" (Michael). The United States Federal Corporation Income Tax rate is graduated, and we chose an effective rate of 34%, which would correspond to a corporation with income between $335,000-$15,000,000 (U.S. Internal Revenue Service). So the combined Federal and state tax rates in the model equal 43.8%.

The electricity PTC was set to $0.022/kWh (electric), which was its value in 2010 (Cooper). We chose the 2010 PTC value in order to remain consistent with all the other adjustments made to reflect real 2010 prices, rather than nominal values. Since the entire model is run using real prices, the PTC does not increase over time (in practice, the PTC includes an annual adjustment equal to the inflation rate). The PTC lasts for the first 10 years of the project. When examining the various possible scenarios, we conduct sensitivity analysis using *no* PTC, and report those results as well. This enables us to

[11] The inflation rate adjustment, *h*, came from the 2014 Economic Report of the President, Table B-3, showing the percent change in the Gross domestic purchases price index

evaluate the value of the PTC, especially since the U.S. Congress has allowed it to lapse on several occasions (sometimes restoring it retroactively).

The Power Purchase Agreement (PPA) rate (which is the rate reimbursed to the electricity producer by the electric company) was set to $0.106/kWh (electric). We chose this rate as our baseline starting point, because it is equal to the rate received by the Laurentian Energy Authority for a somewhat similar (though larger) CHP system in nearby Hibbing and Virginia, Minnesota (Butcher). The Laurentian Energy Authority received a 20-year contract with Xcel Energy to provide electricity from biomass-powered CHP at a PPA rate of $0.102/kWh, which began operation on December 31, 2006. Adjusted for inflation[12] to 2010 dollars (to remain consistent), this is equal to $0.106/kWh. This rate is arguably higher than the market value of the electricity, being more than double what it costs to produce electricity from other fuels (Butcher). However, the purchases help Xcel Energy meet the state's renewable electricity mandates. The PPA rate is often negotiated on a case-by-case basis, so there is not a guarantee that all biomass-electricity producers in Minnesota would receive the same rate for their electricity. This is therefore one of the key factors in the model upon which we conduct sensitivity testing to find the minimum acceptable rate for a producer given a variety of scenarios.

The model contains prices for four forms of energy: TBB chips and wood pellets (the inputs) as well as Ely's heat portfolio and electricity (the outputs). In upcoming sections we will explain several different methods of calculating the future prices for the

[12] The inflation adjustment again came from the 2014 Economic Report of the President, Table B-3, showing the percent change in the Gross domestic purchases price index.

outputs (heat and electricity). To maintain consistent assumptions, we also use the annual prices for the outputs (heat and electricity) to calculate the prices for the two inputs (wood pellets and TBB). The primary physical input for TTB chips is the wood, which is a waste product with little value. The value of the waste product itself is unlikely to change. Instead, transportation is the more significant portion of the cost of TBB at the plant, representing about 71% of its total cost (see Section 3.2 and Table 5.3 in Dovetail Partners "Forest Biomass Heating and Electricity in Cook County, MN"). We do not explicitly develop a cost projection for the diesel fuel that would be required to transport the TBB to the district heating facility. However, because they are both petroleum products, the prices of diesel fuel and heating oil are very strongly correlated (since 1994, their prices have had a correlation of 99.9%, see Figure 9). Heating oil is also the primary component of Ely's heat portfolio, comprising more than ¾ of it. To project future TBB chip prices, then, we adopt the same annual mean reverting percentage change used in Ely's heat portfolio, and apply it to the *transportation portion* of the TBB chip price (71% of the total cost).

In this region of Minnesota, delivery represents about 28.3% of the overall cost of delivered pellets in year zero ($65 of $230 total for delivered pellets). The price of wood pellet production also involves electricity (4.6% of the delivered pellet price) and heating (11.5% of the delivered pellet price) (Dovetail Partners 20). The rest of the pellet production costs are raw materials for which we again assume a constant real-dollar value. We again adopt the progression of heat portfolio prices to represent the progression of diesel fuel for the delivery cost portion of pellet prices, and the electricity price progression is used for the electricity portion of the pellet prices. In the United States the heat for pellet production usually comes from pellet dust or hog fuel. We therefore adopt the progression of TBB prices as the price change for the heating portion of pellet prices. For the purposes of calculating the confidence interval surrounding the log of pellet prices, it is important to remember that the sigmas around the log of heat and electricity prices already represent a joint probability distribution (this is the output of the Dual-Factor Heteroskedstic Stochastic Diffusion that we explained earlier). Therefore, to calculate the confidence interval surrounding the log of pellet prices, we do not need to generate another joint distribution. To find the percentage confidence interval around the expected pellet price, we only need to sum the percentage price confidence intervals of the individual components and then weight them according to the percentage of the pellet price that is comprised by heat and electricity (respectively).

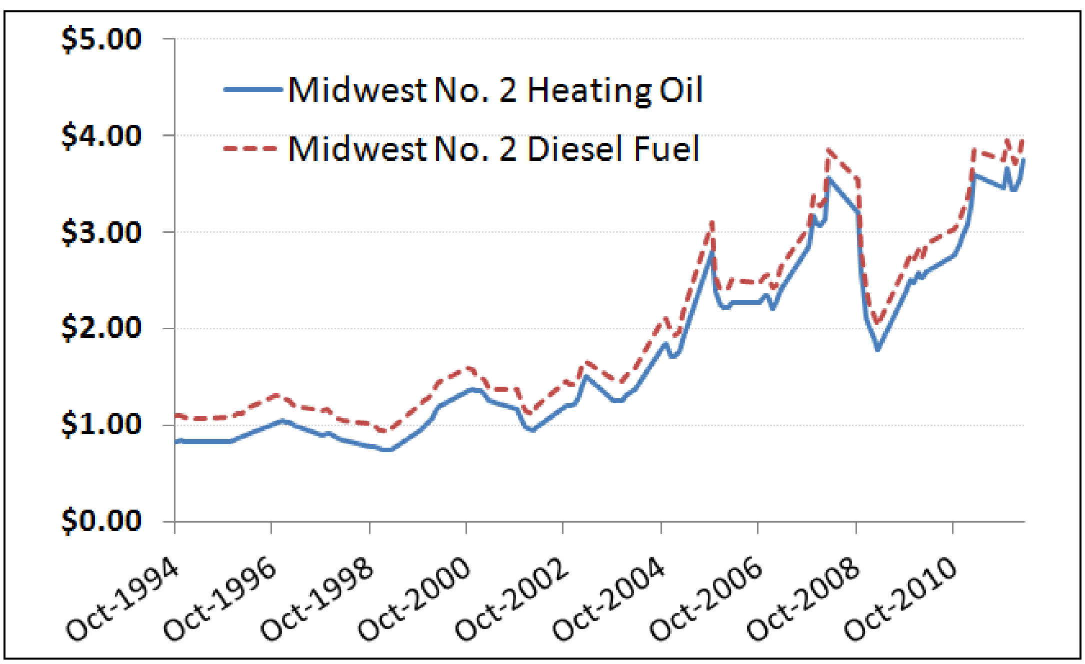


Figure 9: Correlation of Heating Oil and Diesel Fuel Prices (99.9%)

C. Future Expected Price and Volatility when Mean-Reversion is Ignored

We are creating dual-factor 20-year price series for the two saleable commodities of a CHP system: electricity and heat. We have already noted that in Section 1(B) that Dixit and Pindyck propose that commodity prices can be mean-reverting. The importance of acknowledging Mean Reversion is also noted by Laughton and Jacoby in their article "Reversion, Timing Options, and Long-Term Decision-Making." In that article they demonstrate that ignoring mean reversion can result in biased project valuations. The bias is particularly problematic when considering long-term investment alternatives, such as 20-year projects, because uncertainty in later phases of the project may be greatly overstated. Figures 10 and 11 use actual historical heating prices beginning in 1990 to demonstrate this uncertainty. Over this time period, the natural logs of the prices experienced a 20-year annual upward drift of 0.7% with an annual volatility (σ) of 5.5%. (In these Figures, we adopt Dixit and Pindyck's convention of shading the 66% confidence interval because it represents +/- one standard deviation from the expected price.) Compounded over 20 years, the volatility of the Geometric Browning Motion process would mean there is only a 66% chance that the heat price in year 2010 would be under $70/mmBtu. In contrast, however, using a mean-reverting approach would narrow the uncertainty throughout, so that in year 2011 it would be 66% likely that the price of heat would be under $38/mmBtu. This price is much lower, and it is more reflective of the *actual* price walk over that period.

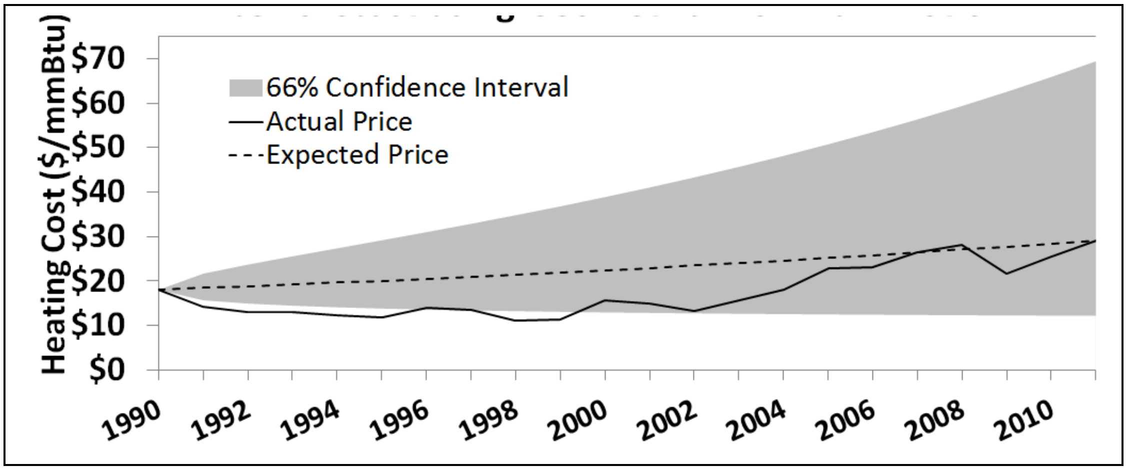


Figure 11: Heating Cost Price Forecast Using Geometric Brownian Motion (GBM)

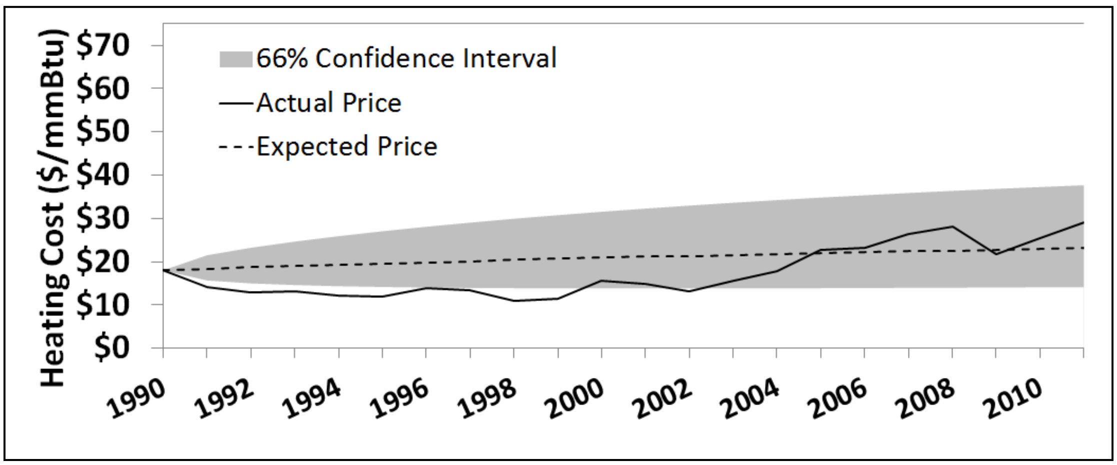


Figure 10: Heating Cost Price Forecast using a Mean Reverting Process

The problem with using the volatility from Figure 10 in a Cash Flow model of a future DH/CHP system is that the uncertainty in Year 20 is so high that we may obtain unrealistic outputs for the system's projected financial performance. This is not only a problem for traditional Discounted Cash Flow analysis; Hahn and Dyer note that when this uncertainty is included in a Real Options investment analysis, it can result in an unrealistically high value for the option volatility (see Section 4 for a more thorough explanation of Real Options analysis). As a result, the value of the option can appear unrealistically high. In subsequent sections we will explain the Real Options in more detail, but as a preliminary explanation of the impact of Mean Reverting prices, we present Figures 12 and 13, which compare the option volatility using Mean Reverting Prices and without using Mean Reverting prices. The volatility (measured as the standard deviation of the distribution) is much higher in the version of the model without mean reverting prices, and it even indicates that in 2.5% of the scenarios, the value of the option exceeds $50 Million, and has a maximum value of $312 Million. This excessively high option volatility could lead to an overestimation of the option value, and (possibly) to a misdirected investment, which is the danger that Hahn and Dyer warn against. When we apply the techniques of Real Options analysis in Section 4, we will present the Real Options results once using GBM expected prices and once using mean-reverting expected prices to demonstrate this difference.

D. Future Expected Price and Volatility Calculation Using Mean Reversion

The first step in this process is to calculate the historical mean reversion parameters for each of these commodities using the techniques from Section 2.E. For the historical electricity data we used information from the EIA that tracked Minnesota residential electricity prices since 1990 (U.S. Energy Information Administration "Form EIA-826 detailed data"). We converted these nominal prices into real 2010 prices using the annual gross domestic product figures from the National Income and Product Accounts Table 1.1.9. (Implicit Price Deflators for Gross Domestic Product). This is

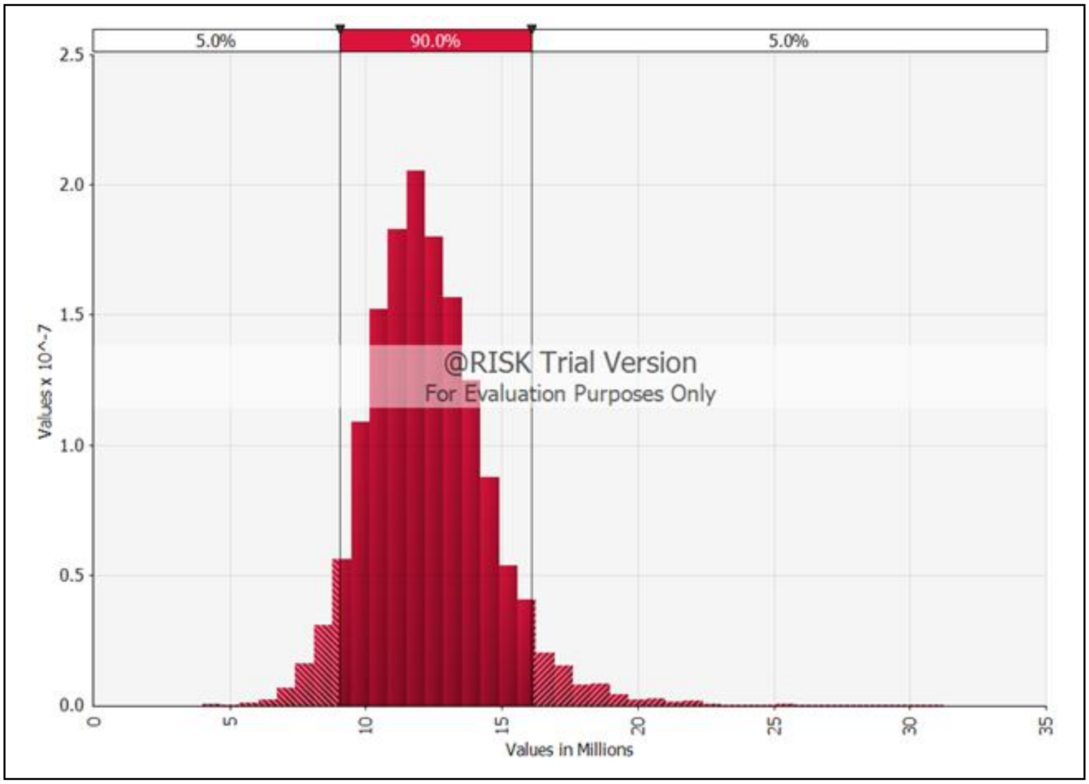


Figure 13: Present Value of Option Investment with Mean Reverting Prices

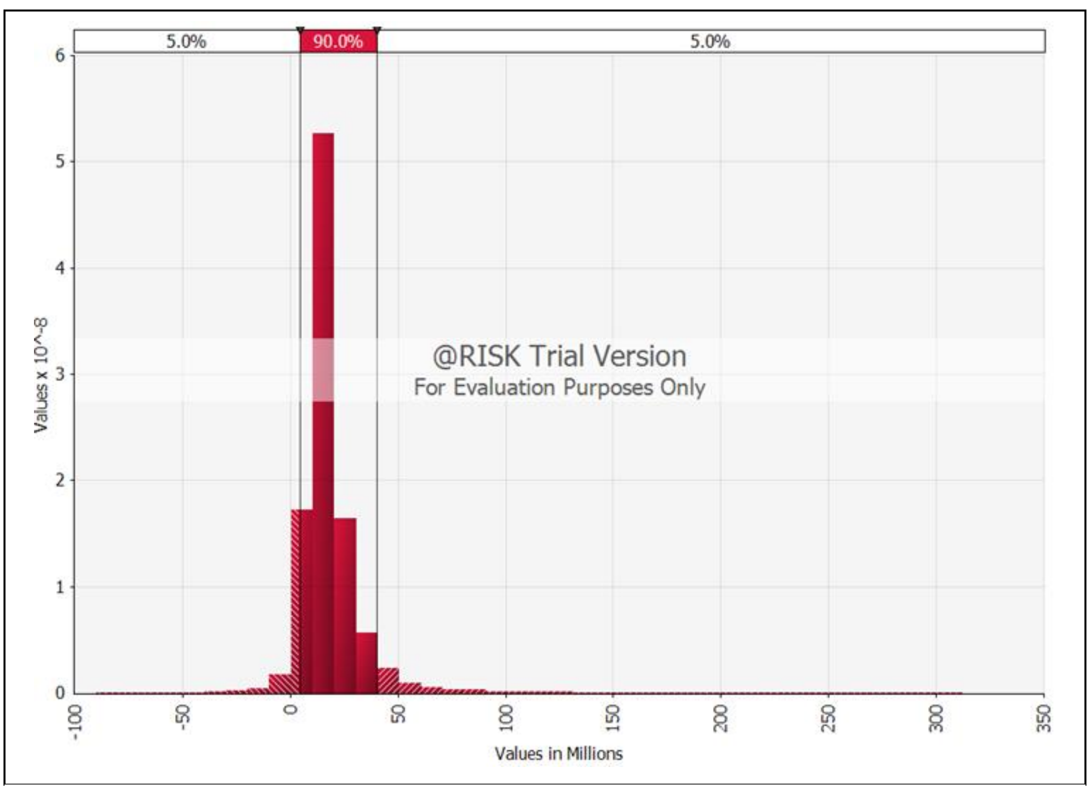


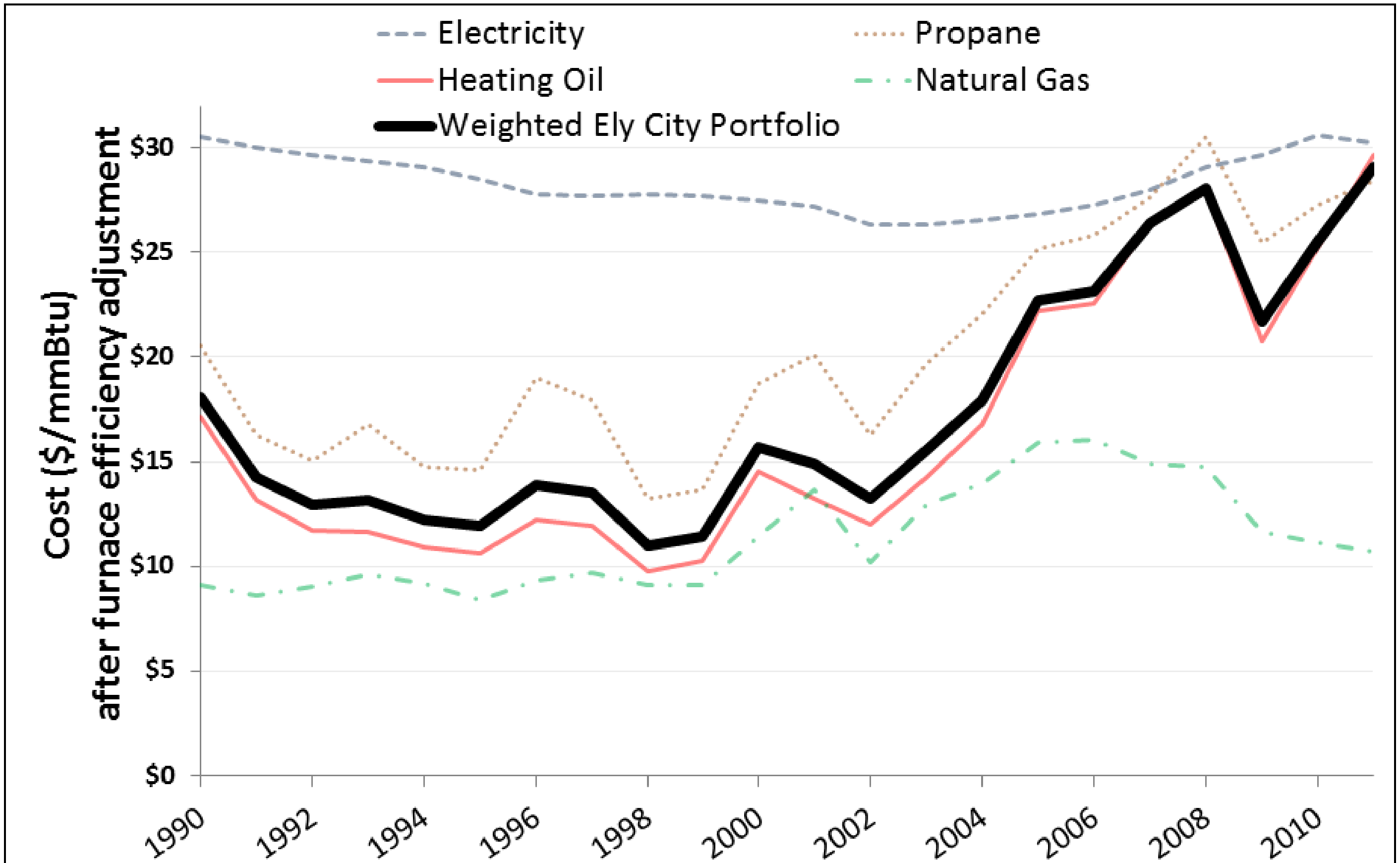
Figure 12: Present Value of Option Investment without Mean Reverting Prices

published quarterly by the Bureau of Economic Analysis, and then converted them to a mmBtu basis. The results can be seen as the "Electricity" line on Figure 14.

Calculating the historical mean reversion parameters for heat in Ely involved the additional step of creating a multi-fuel heating "portfolio" for the town. According to the year 2000 census (see Figure 16), the most common heating fuel used in the Ely area was Heating Oil/Kerosene, followed by propane (aka: LPG). This is significant because these are both relatively expensive fuels when compared to natural gas[13] (see Figure 14). Unlike many urban areas, the town of Ely does not have convenient access to a of natural gas delivery system, which is unfortunate because natural gas is currently much less expensive than other fuels.

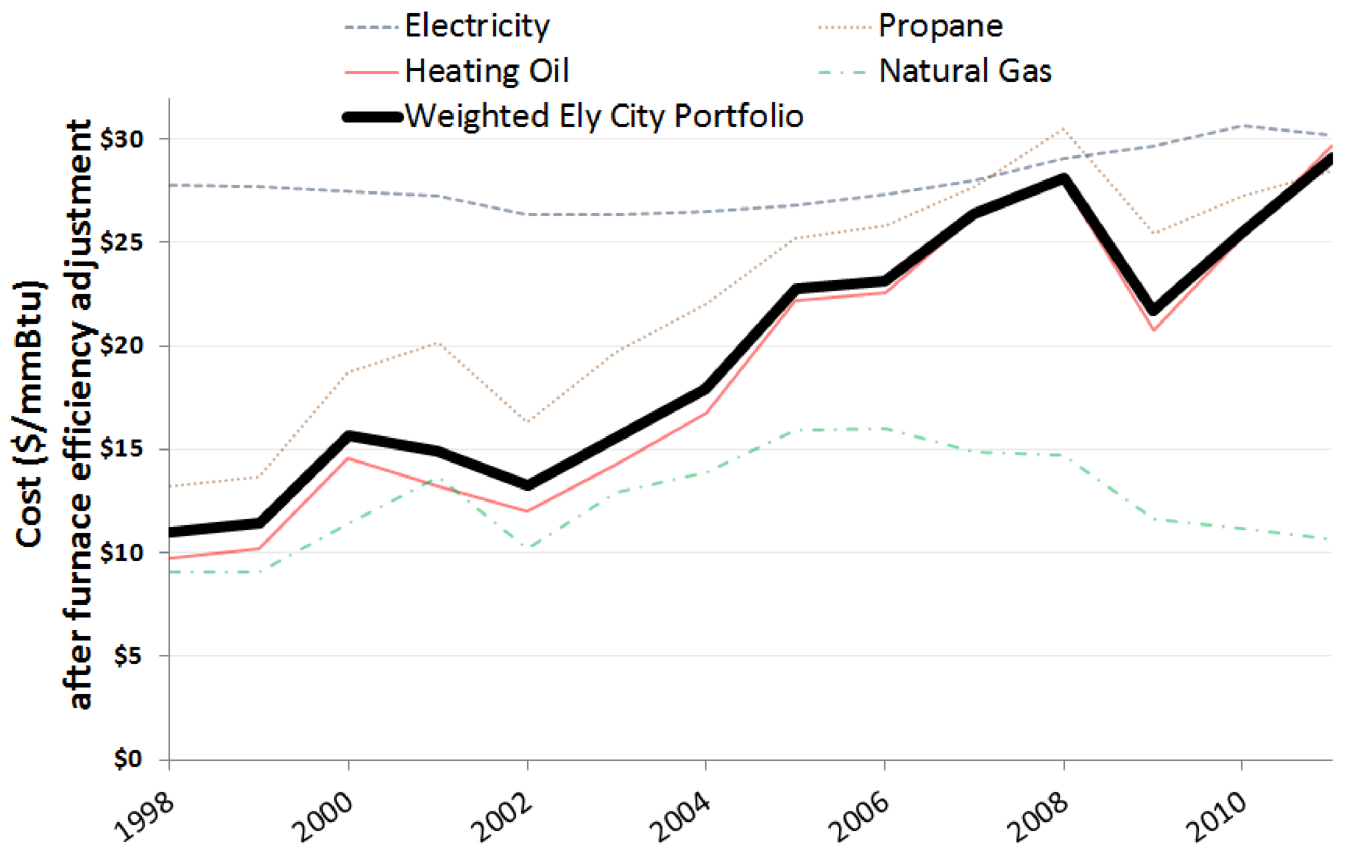
When conducting the regression solve for the mean reversion parameters that is described in Section 2(E), the starting year is important. We have chosen to use prices starting in 1990 for two reasons: first, because the life of the DE project is 20 years, we wanted to have a dataset that was also 20 years or more; second, we are limited in how far back the Energy Information Administration (EIA) publishes consistent historical prices for heating oil and propane. These go back only as far as 1990, so these are the limiting data. However, if we had chosen to begin the analysis starting in 1998, the heat price progression would probably not have appeared to be mean-reverting (with the exception, perhaps of natural gas). To see this, compare the trend of the Weighted Ely City Portfolio in Figure 14 that begins in 1990 to Figure 15, which begins in 1998. Beginning in 1998, the trend appears to be consistently upward. This demonstrates the

[13] Note: In the Census data, Natural Gas is called "Utility Gas."



Source: EIA Data for Minnesota or the Midwest

Figure 14: Historical Minnesota Residential Heating Fuel Costs and Portfolio Price (constant 2010 Dollars), beginning in 1990



Source: EIA Data for Minnesota or the Midwest

Figure 15: Historical Minnesota Residential Heating Fuel Costs and Portfolio Price (constant 2010 Dollars), beginning in 1998

importance of having a justification for the starting period and for using a large quantity of time periods in a mean-reversion analysis. Dixit & Pindyck discuss this issue in depth in regard to copper prices: if using data from 120 years, the prices are clearly mean-reverting (if slowly). But with data from only 30-40 years, it is not statistically possible to definitively reject the “random walk” hypothesis in favor of mean-reversion (pp. 77-78).

Using the 22 years of data we have for heating and electricity prices in Ely, we create a weighted portfolio of fuels (the historical price of which is also shown in Figure 14) based on data from the 2000 Census Summary File, Profile of Housing characteristics for Ely City, and then we calculate the mean reversion parameters for Ely’s heating costs based on this portfolio. Disregarding the 3% of homes that use alternative fuels, the portfolio consists of 78% heating oil, 16% propane, 4% electric and 2% natural gas (see Figure 16). For historical prices we use data from the Energy Information Administration (EIA). We used annual average nominal residential Minnesota prices for heating oil, propane, electricity (U.S. Energy Information Administration “Form EIA-826 detailed data”), and natural gas (U.S. Energy Information Administration “Minnesota Price of Natural Gas Delivered to Residential Consumers”), which we then converted to a mmBtu basis (For conversion factors, see: National Energy Information Center “Heating Fuel Comparison Calculator”) All of the fuel prices except electricity were divided by a furnace efficiency factor of 78%. This is because in terms of usable Btu’s, electricity is nearly 100% efficient, while the other fuels are dependent upon furnace conversion efficiencies(National Energy Information Center “Heating Fuel Comparison Calculator”) usually around 78%. Again, we also converted the nominal prices into real 2010 prices

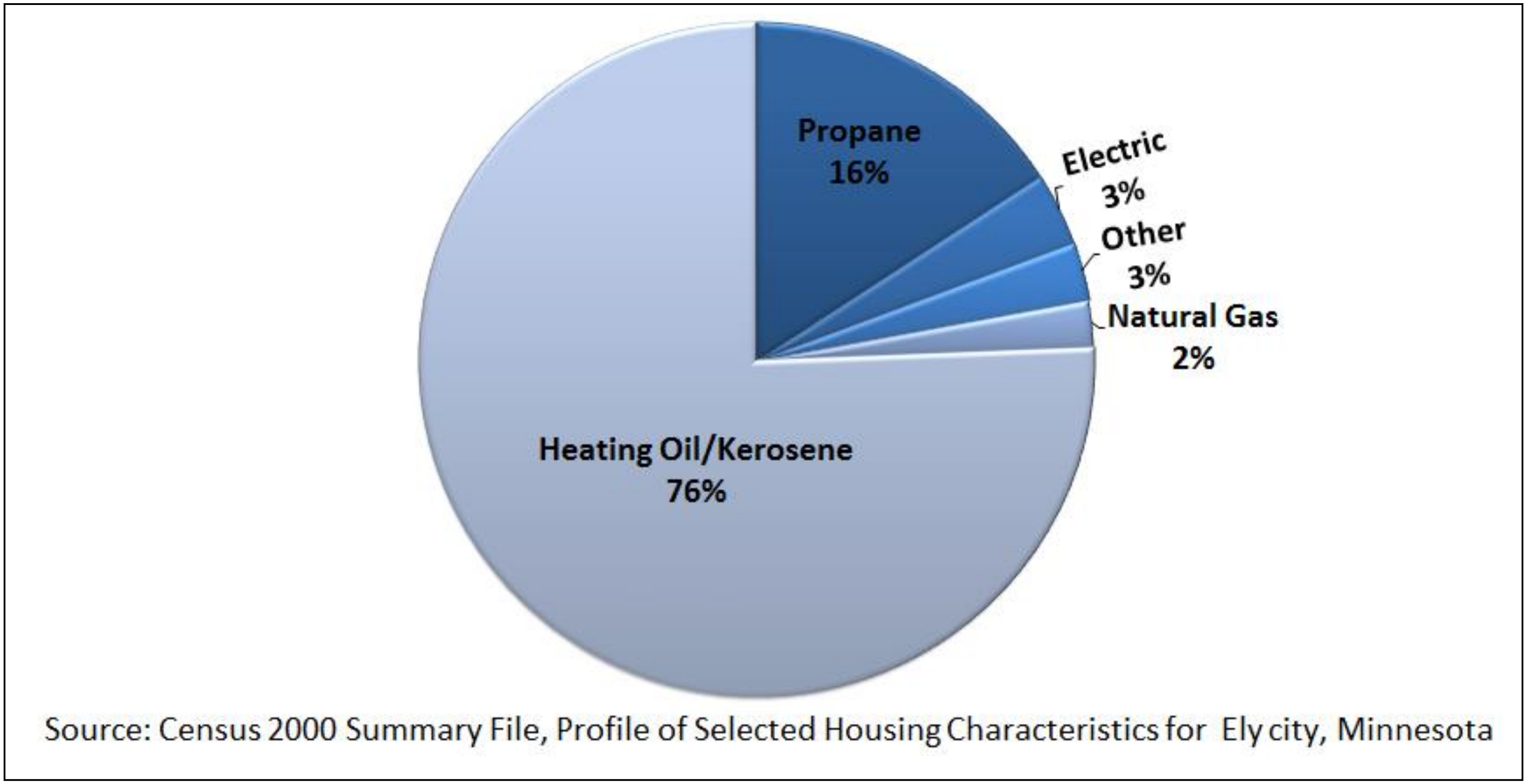


Figure 16: House Heating Fuel Portfolio for Ely, Minnesota (percent of structures)

using the annual gross domestic product figures from the National Income and Product Accounts Table 1.1.9. (Implicit Price Deflators for Gross Domestic Product). This is published quarterly by the Bureau of Economic Analysis.

Using the technique detailed in Section 2(E), we develop the mean reversion parameters for electricity and heat, and these are displayed in Table 4. The parameters tell us that we are currently experiencing high energy prices for both electricity and heat, and that over the next 20 years, these prices would be expected to revert closer to their mean reversion price. Electricity is currently 7.8% more expensive than its mean reversion price, while Ely's heat portfolio is currently 8.7% more expensive.

	Electricity	***Heat***
Current Price (X_0) (\$/mmBtu)	*\$30.26*	*\$29.10*
$\hat{a}$	0.3079 (p = *0.29*)	0.1504 (p = *0.66*)
$\hat{b}$	-0.0923 (p = *0.29*)	-0.0458 (p = *0.70*)
Mean Reverting Price ($\bar{X} = e^{-\hat{a}/\hat{b}}$) (\$/mmBtu)	\$28.06	\$26.78
Reversion Speed ($\hat{\eta}$)	0.0969	0.0468
Volatility ($\hat{\sigma}$)	1.972%	16.149%
Correlation of Prices ($\hat{\rho}$)	22.294%	

Table 4: Mean Reversion Parameters for Electricity and Heat Portfolio in Ely, Minnesota

The *p*-values on the $\hat{a}$ and $\hat{b}$ coefficients (which are used to calculate the mean-reversion price) are not statistically significant. As described by Dixit and Pindyck, this could be because the mean reversion is too slow to statistically differentiate from a random walk when only using 22 data points. However, looking again at the heat portfolio prices over the last 22 years shown in Figures 10 and 11, we can graphically show that the mean-reverting price walk and confidence intervals are more reflective of the actual prices than Geometric Brownian Motion (GBM) is. This is especially true in the later years, when the expected prices under GBM become quite high, and the 66% confidence interval around them extends far above the level of observed prices.

There are two ways we deal with the fact of the coefficients on $\hat{a}$ and $\hat{b}$ are not statistically significant. First, we conduct all of the analyses (with the exception of the real options sensitivity testing) using both the non-mean-reverting Geometric Brownian Motion price model *as well as* the mean-reverting price model. If prices are not genuinely mean-reverting, or they are mean-reverting at a slower rate, then the "truth" lies between the two price models. We will see that using expected energy prices from *non*-mean-reverting price model results in higher performance measures, but much higher volatility around those measures. The second way we deal with the non-statistically significant coefficients on $\hat{a}$ and $\hat{b}$ is to perform sensitivity testing by exaggerating and dampening the annual price changes in the mean-reverting model. We do this in Part C. We double the annual change in prices, and we cause prices to change by the expected percentage but in the opposite direction. In between those extremes, we also include one scenario in which expected prices do not change at all over the next 20 years. We chart the results, making it possible to interpolate for the expected outcome of many different scenarios.

Using the mean reversion parameters solved for in Table 4, we employ the techniques described in Section 2(D) in conjunction with our original Visual Basic program to solve in each time step (that is, annually) for the expected prices and price distributions of the various energy fuels that contribute to the model. These fuels are: on the input side, wood pellets and top & branch biomass (TBB) chips. On the output side the fuels are electricity and Ely's "heat portfolio."

We are conducting a 20-year annual cash flow analysis of the CHP system, so we use annual data to develop the mean reversion parameters and output annual average expected prices and the distribution around them. Tables 5 and 6 on the following pages show the 20-year series for electricity and heat, and the probabilities associated with each price and time period. With the mean reverting model, the per-mmBtu expected prices of both electricity and heat drift lower toward their mean reversion. The accompanying graphs show the differences between the expected prices and the standard deviations around those prices when solving the time series using a GBM process compared to a mean-reverting process (for electricity prices, Figures 17 and 18; for heat prices, Figures 19 and 20). It is abundantly clear that solving using a mean reverting process provides a much narrower range of potential future prices. Particularly for heating prices, the mean

Time Period:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
X Expected Value without Mean Reversion:	$30.26	$30.24	$30.23	$30.21	$30.19	$30.17	$30.16	$30.14	$30.12	$30.11	$30.09	$30.07	$30.05	$30.04	$30.02	$30.00	$29.99	$29.97	$29.95	$29.93	$29.92
X Expected Value with Mean Reversion:	$30.26	$30.03	$29.83	$29.65	$29.48	$29.34	$29.21	$29.09	$28.98	$28.88	$28.80	$28.72	$28.65	$28.59	$28.53	$28.48	$28.43	$28.39	$28.35	$28.32	$28.29
X Std Dev of Log(Price) without Mean Reversion:	$0.00	$0.02	$0.03	$0.03	$0.04	$0.04	$0.05	$0.05	$0.05	$0.06	$0.06	$0.06	$0.07	$0.07	$0.07	$0.07	$0.08	$0.08	$0.08	$0.08	$0.09
X Std Dev of Log(Price) with Mean Reversion:	$0.00	$0.02	$0.03	$0.03	$0.03	$0.04	$0.04	$0.04	$0.04	$0.04	$0.04	$0.04	$0.04	$0.04	$0.04	$0.04	$0.04	$0.04	$0.04	$0.04	$0.04
$20.40																					0%
$20.80																				0%	
$21.22																			0%		0%
$21.64																		0%		0%	
$22.07																	0%		0%		0%
$22.51																0%		0%		0%	
$22.96															0%		0%		0%		0%
$23.42														0%		0%		0%		0%	
$23.88													0%		0%		0%		0%		0%
$24.36												0%		0%		0%		0%		0%	
$24.84											0%		0%		0%		0%		0%		0%
$25.34										0%		0%		1%		1%		1%		2%	
$25.84									0%		1%		2%		3%		3%		4%		5%
$26.36								1%		3%		5%		6%		8%		9%		10%	
$26.88							3%		7%		10%		12%		14%		16%		17%		18%
$27.42						7%		13%		17%		21%		23%		25%		26%		27%	
$27.96					14%		22%		27%		30%		31%		32%		33%		34%		34%
$28.52				26%		34%		36%		37%		37%		36%		36%		35%		35%	
$29.09			44%		45%		42%		39%		37%		35%		33%		31%		30%		29%
$29.67		69%		51%		41%		35%		31%		28%		25%		24%		22%		21%	
$30.26	100%		48%		34%		27%		22%		19%		16%		15%		13%		12%		11%
$30.86		31%		21%		16%		13%		10%		9%		8%		7%		6%		5%	
$31.48			8%		7%		6%		5%		4%		3%		3%		2%		2%		2%
$32.10				2%		2%		1%		1%		1%		1%		1%		1%		1%	
$32.74					0%		0%		0%		0%		0%		0%		0%		0%		0%
$33.40						0%		0%		0%		0%		0%		0%		0%		0%	
$34.06							0%		0%		0%		0%		0%		0%		0%		0%
$34.74								0%		0%		0%		0%		0%		0%		0%	
$35.43									0%		0%		0%		0%		0%		0%		0%
$36.14										0%		0%		0%		0%		0%		0%	
$36.86											0%		0%		0%		0%		0%		0%
$37.59												0%		0%		0%		0%		0%	
$38.34													0%		0%		0%		0%		0%
$39.10														0%		0%		0%		0%	
$39.88															0%		0%		0%		0%
$40.68																0%		0%		0%	
$41.49																	0%		0%		0%
$42.32																		0%		0%	
$43.16																			0%		0%
$44.02																				0%	
$44.89																					0%

Table 5: 20-Year Price Probability Tree for Electricity ($/mmBtu)

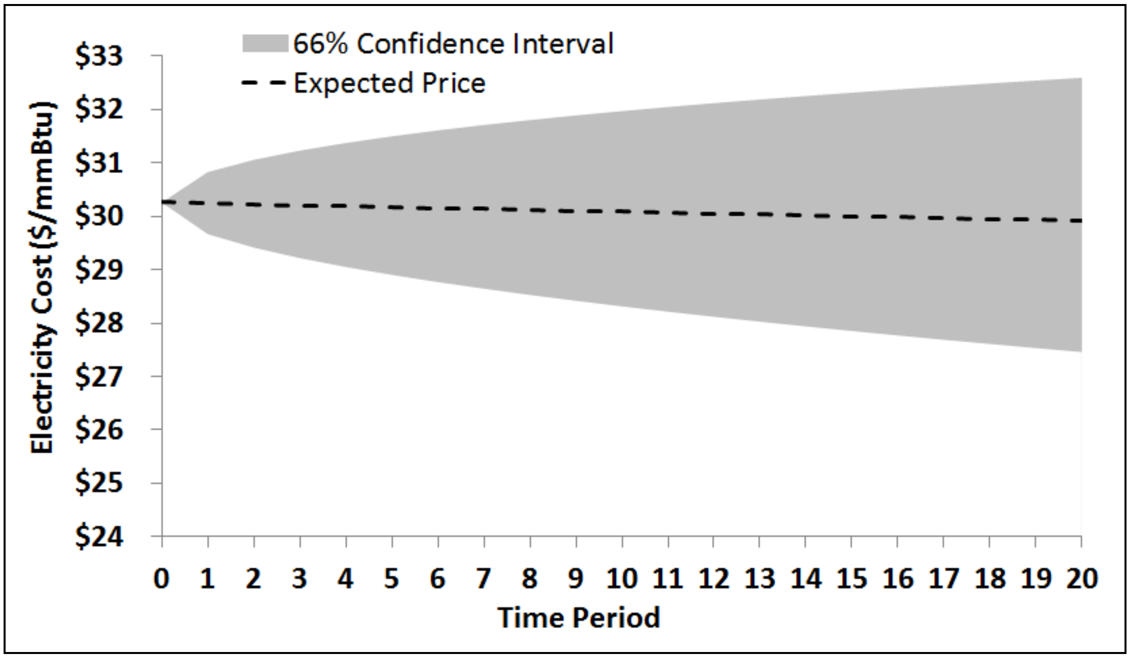


Figure 17: 20-Year Electricity Price Forecast Using GBM

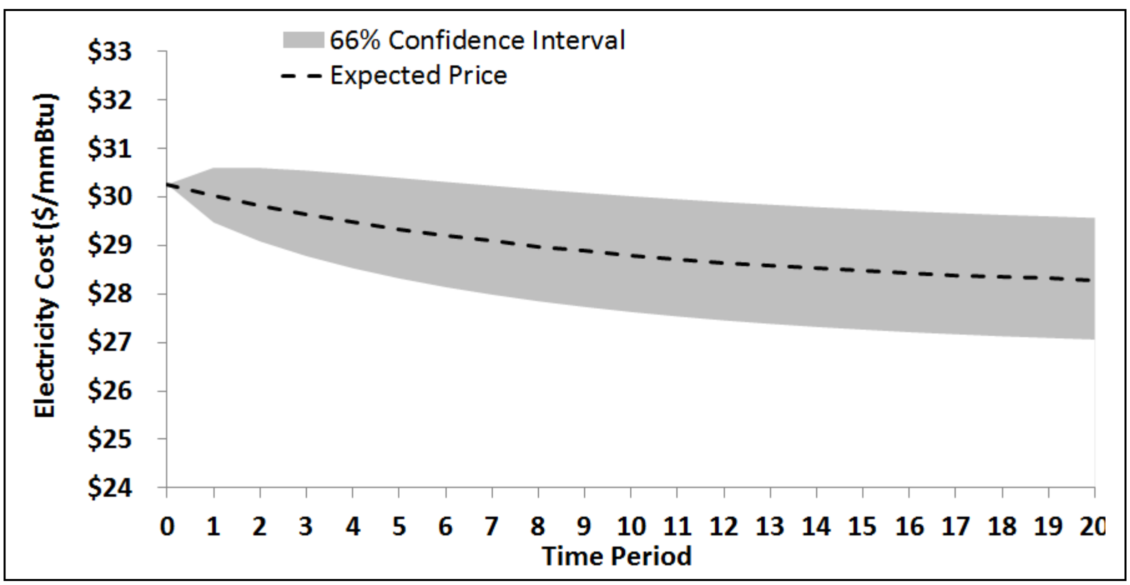


Figure 18: 20-Year Mean-Reverting Electricity Price Forecast

Time Period:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Y Expected Value without Mean	$29.10	$29.83	$30.57	$31.35	$32.15	$32.97	$33.83	$34.71	$35.62	$36.56	$37.53	$38.54	$39.58	$40.66	$41.78	$42.93	$44.13	$45.36	$46.65	$47.97	$49.35
Y Expected Value with Mean Reversion:	$29.10	$28.61	$28.15	$27.72	$27.32	$26.94	$26.58	$26.25	$25.93	$25.64	$25.35	$25.09	$24.84	$24.61	$24.38	$24.18	$23.97	$23.79	$23.61	$23.44	$23.28
Y Std Dev of Log(Price) without Mean	$0.00	$0.19	$0.27	$0.33	$0.39	$0.43	$0.48	$0.52	$0.56	$0.60	$0.64	$0.68	$0.71	$0.75	$0.78	$0.82	$0.85	$0.88	$0.92	$0.95	$0.98
Y Std Dev of Log(Price) with Mean Reversion:	$0.00	$0.16	$0.22	$0.26	$0.29	$0.32	$0.35	$0.37	$0.38	$0.40	$0.41	$0.42	$0.43	$0.44	$0.45	$0.46	$0.46	$0.47	$0.48	$0.48	$0.49
$1.15																					0%
$1.35																				0%	
$1.59																			0%		0%
$1.87																		0%		0%	
$2.20																	0%		0%		0%
$2.58																0%		0%		0%	
$3.03															0%		0%		0%		0%
$3.57														0%		0%		0%		0%	
$4.19													0%		0%		0%		0%		0%
$4.92												0%		0%		0%		0%		0%	
$5.79											0%		0%		0%		0%		0%		0%
$6.80										0%		0%		0%		1%		1%		1%	
$7.99									0%		1%		1%		1%		2%		2%		2%
$9.40								1%		1%		2%		3%		3%		4%		5%	
$11.04							1%		3%		4%		5%		6%		7%		8%		8%
$12.98						3%		6%		8%		9%		11%		11%		12%		13%	
$15.25					7%		11%		13%		15%		16%		17%		17%		18%		18%
$17.93				15%		19%		21%		22%		22%		22%		23%		23%		23%	
$21.07			29%		30%		30%		29%		28%		27%		27%		26%		26%		26%
$24.76		55%		43%		37%		34%		31%		30%		29%		28%		27%		26%	
$29.10	100%		52%		40%		35%		31%		29%		27%		26%		25%		24%		24%
$34.20		45%		35%		30%		27%		25%		23%		22%		21%		20%		20%	
$40.19			19%		19%		19%		18%		17%		16%		16%		15%		15%		14%
$47.24				8%		10%		10%		10%		10%		10%		10%		10%		9%	
$55.52					3%		4%		5%		5%		6%		6%		6%		6%		5%
$65.25						1%		2%		2%		3%		3%		3%		3%		3%	
$76.68							0%		1%		1%		1%		1%		1%		1%		1%
$90.12								0%		0%		0%		0%		0%		1%		1%	
$105.92									0%		0%		0%		0%		0%		0%		0%
$124.48										0%		0%		0%		0%		0%		0%	
$146.30											0%		0%		0%		0%		0%		0%
$171.94												0%		0%		0%		0%		0%	
$202.08													0%		0%		0%		0%		0%
$237.50														0%		0%		0%		0%	
$279.12															0%		0%		0%		0%
$328.04																0%		0%		0%	
$385.54																	0%		0%		0%
$453.11																		0%		0%	
$532.52																			0%		0%
$625.85																				0%	
$735.54																					0%

Table 6: 20-Year Price Probability Tree for Heat ($/mmBtu)

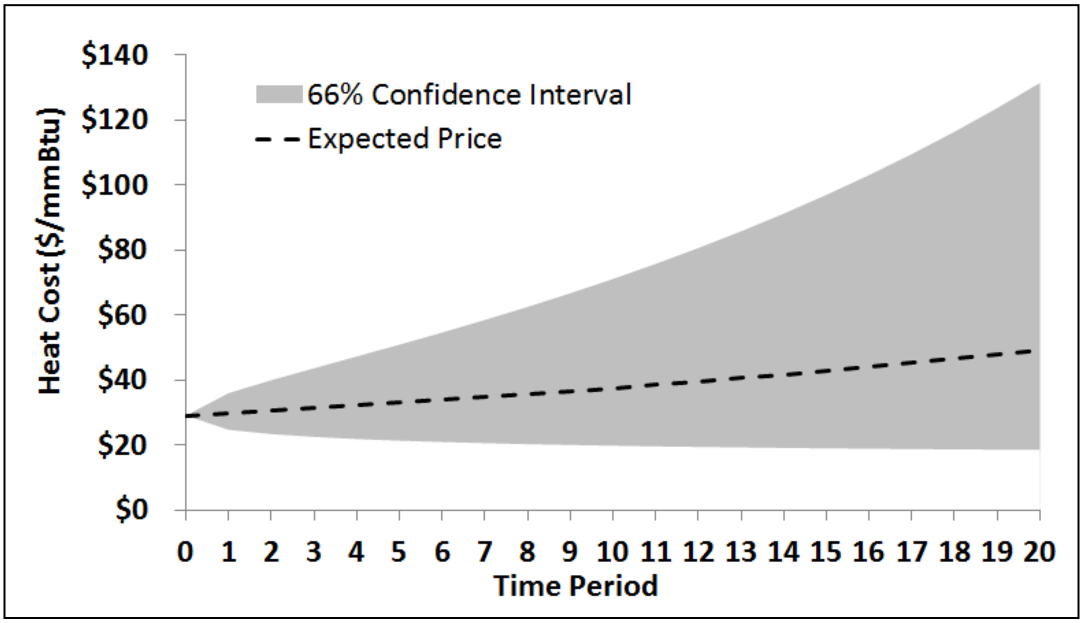


Figure 19: 20-Year Heat Price Forecast Using GBM

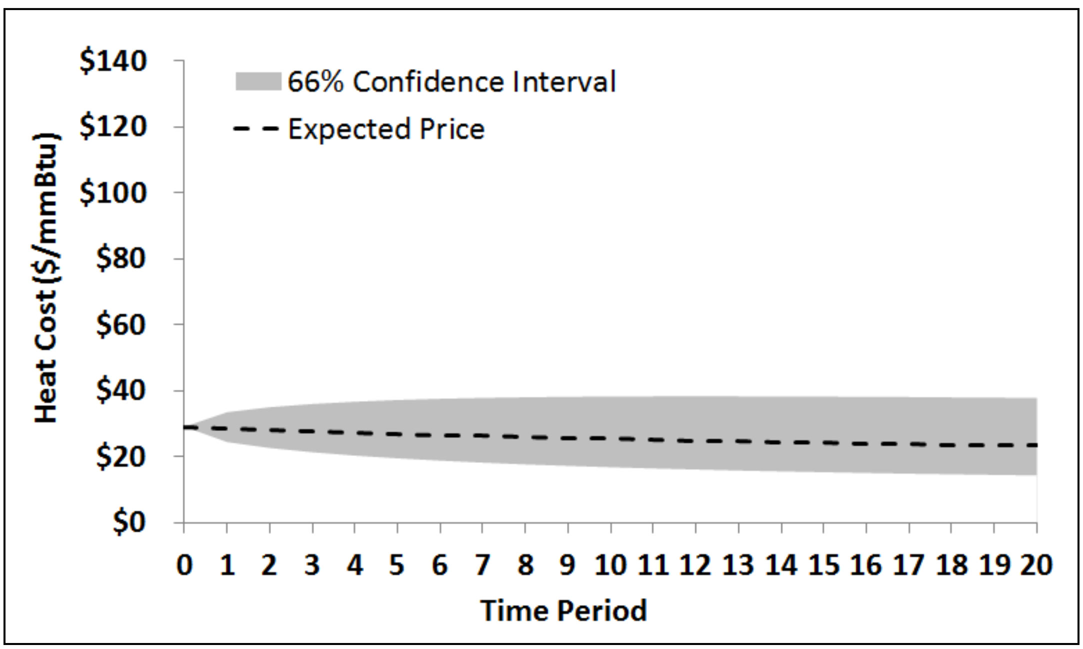


Figure 20: 20-Year Mean-Reverting Heat Price Forecast

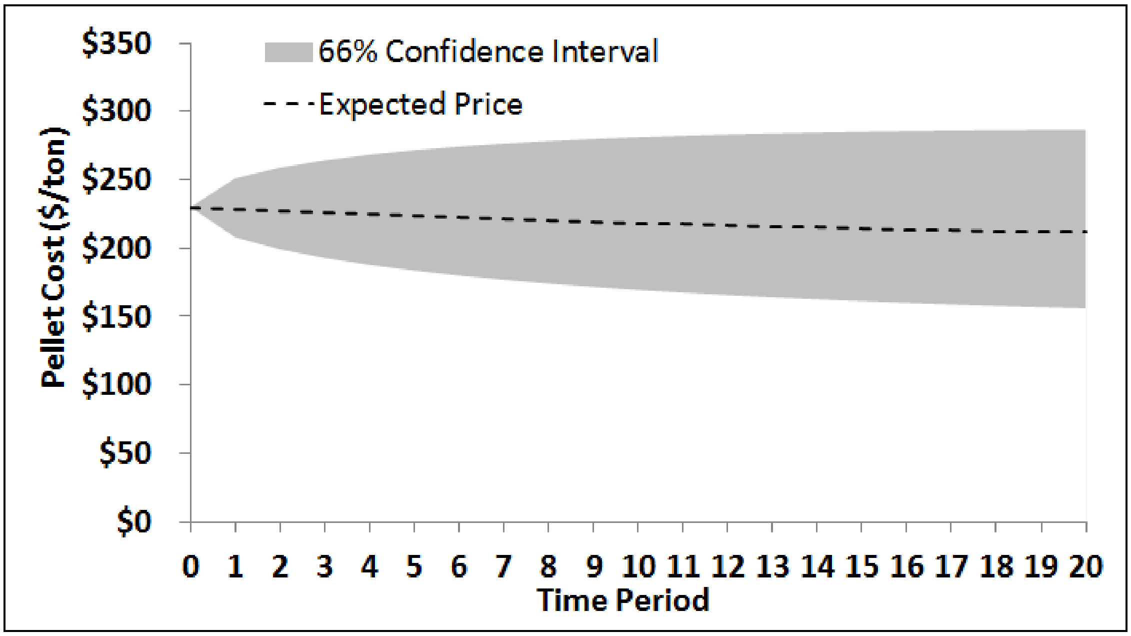


Figure 21: Pellet Price Forecast using a Mean Reverting Process

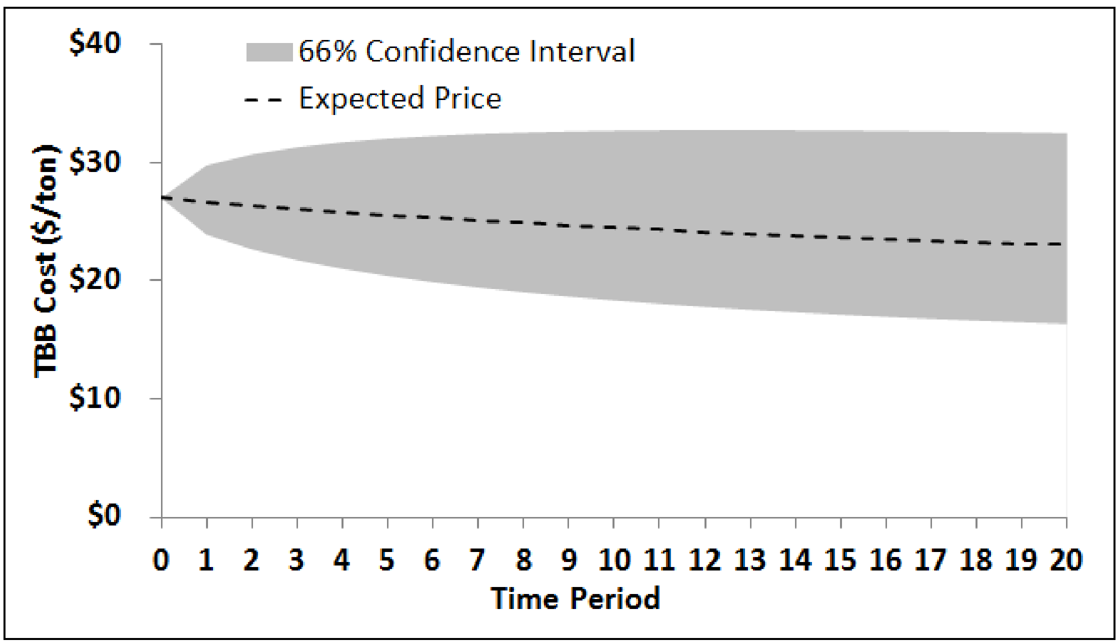


Figure 22: TBB Price Forecast using a Mean Reverting Process

reversion projection also provides a much more reasonable range of possible prices. It would seem unrealistic to say with only 66% confidence that 20 years in the future the price of heat in Ely could be between \$18.50 and \$131.50/mmBtu (in today's dollars). It is rather more likely that before heating prices would rise to that high level, technological changes and demand shifts would pull them downward again. Figures 21 and 22, show the 20-year series for the pellet and TBB input prices, and the 66% confidence interval associated with each price and time period shown in gray. As described in Section 3B, the prices for wood pellets and TBB are calculated based on the prices of heat and electricity in each time period.

E. The Basic Cash Flow Model Using Expected Prices Without and With Mean Reversion

In the simplest version of the model, we can use these input and output prices to develop cash flow and revenue curves. First we examine the complete (non-incremental) project. We separate the analysis into three steps: we first look at it before taxes and financing, then we look at it after taxes & financing (generally, taxes would be a drag on profitability, but in this case some government subsidies come in the form of avoided tax liabilities, which could make this project more attractive than another project that has an equivalent rate of return but on which the profits would be taxed at conventional rates), and finally we apply a discount rate (a.k.a. the hurdle rate of return necessary to attract investment, or the discount rate that accounts for the time value of money).

Using the non-mean-reverting expected prices for electricity and heat shown in Figure 17 and Figure 19, we can calculate the cash flows and performance metrics for the project over its lifetime. These cash flows are shown in Figure 23, with the most relevant line being the bold "Cumulative Net Cash Flow (after taxes and financing, discounted)." This line would tell a utility investor that at a discount rate of 4.25%, the project would have a positive cash flow in every year, assuming it is 100% financed at a rate of 4.25% over 20 years. In other words, the project would be profitable at the current hurdle rate of return, assuming prices progress on a non-mean-reverting path.

The "Baseline Scenario" column in Table 7 shows the Baseline Scenario input assumptions for the non-mean-reverting analysis at the top, and the bottom half of the table shows the numerical output for several profitability metrics. The other two columns in Table 7 show the impact of varying some of these assumptions. Microsoft Excel's solver function makes it very easy to solve for the thresholds that would make the project profitable under a variety of assumptions. We call a "profitable" project one in which the after-tax Net Present Value is greater than zero. For example, the PPA rate could go all the way down almost to $0.012/kWh, or the investor could loan (bond) capital at an annual rate of almost 8.56%, and the project would still be considered profitable.

The first fundamental change we make to this model, however, is to show what would happen if expected prices for electricity and heat would instead follow a mean-reverting course over the life of the project. That is, if expected prices follow the paths shown in Figure 18 and Figure 20. In this scenario, the project is technically still "profitable," based on the definition we have set of having a NPV greater than zero, but only slightly so. The after-tax NPV drops from $7,657,060 in the non-mean-reverting model to just $166,646 in the model with mean-reverting prices. This is because after the

PTC and accelerated depreciation expire in year 10, the profits must cover higher taxes as well as continue to service the debt on the project. So after year 10, the distributable cash flow after debt payments and taxes is *negative*. This can be seen in Figure 24, where the dark black line for the "Cumulative Net Cash Flow (after taxes and financing, discounted)" begins to decline. The project's life is 20 years, but if it were 21 years, it would not have a positive NPV. Not only is the value of the project declining in later years, in Table 8 we show that the project can be brought to the edge of profitability by slightly tweaking some of the major underlying assumptions. For instance, if the PPA rate were only *very slightly* lower ($0.104 instead of $0.106), the project would no longer be profitable. In this way, the expectation of profitability is much more marginal (and perhaps risky) when using mean-reverting prices than when using non-mean-reverting (GBM) prices.

F. Introducing a Tax Equity Partner to the Cash Flow Model

The cash flow model is also useful when assessing the impact of the U.S. Federal Government's complex approach to subsidizing renewable energy projects. Unlike Europe, which uses feed-in-tariffs to incentivize Renewable Energy production, the U.S. Federal Government primarily adopts tax equity approaches (Sharif, *et al.*). This can make renewable energy project financing more complex for individuals or small investors who do not have enough tax exposure to fully utilize the tax incentives in all the years that the government offers them. In such cases, the renewable energy industry has developed various arrangements for capturing the full value of tax subsidies. Under these arrangements, the developer of the renewable energy project (who lacks the tax

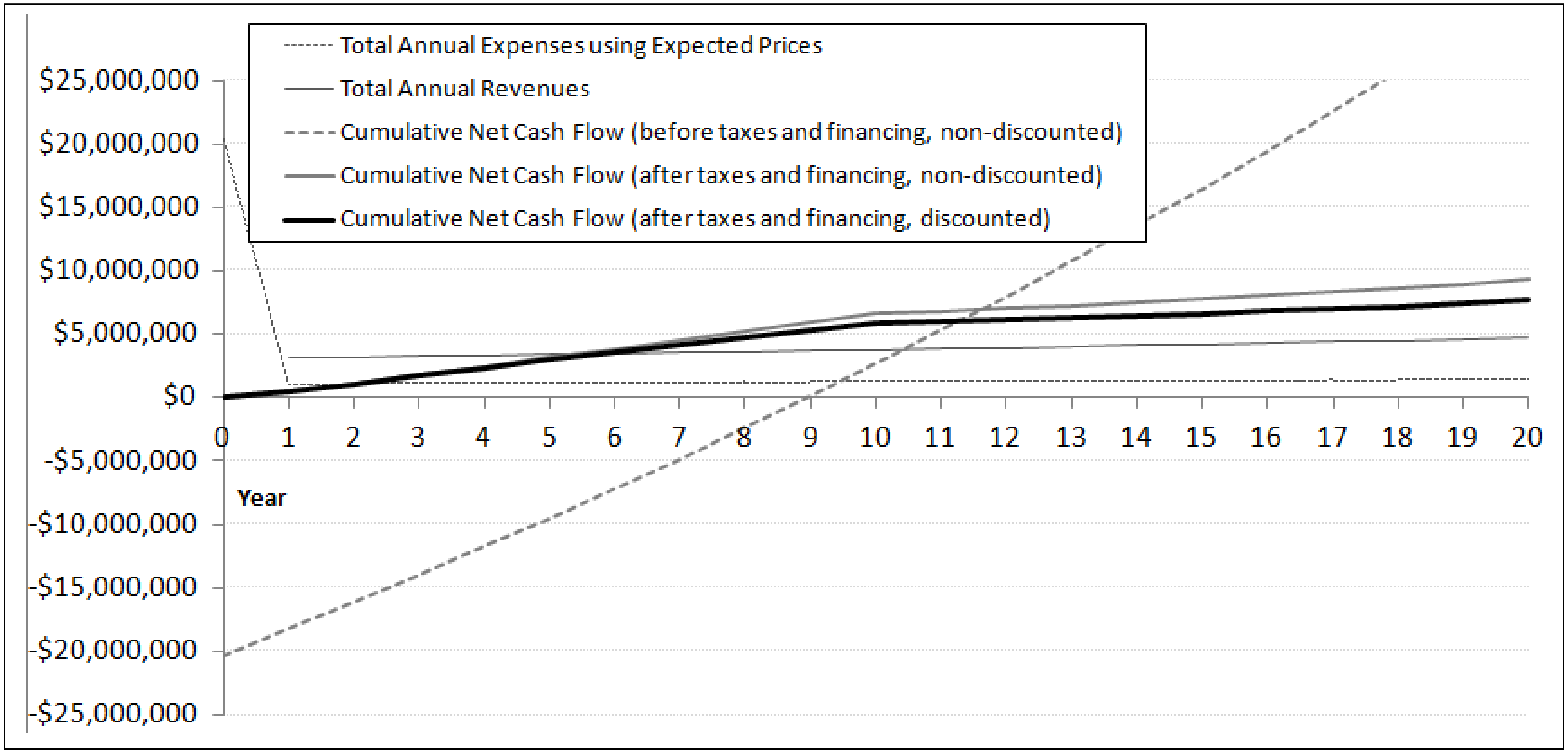


Figure 23: Cumulative Cash Flows of Non-Incremental Project using <u>non-mean reverting</u> expected prices when full tax shield subsidies go unutilized

Scenario Assumptions	Baseline Scenario	Lower PPA Rate	Maximum Interest Rate	No PTC
Useful life	20 years	20 years	20 years	20 years
Years of depreciation	10 years	10 years	10 years	10 years
Hurdle rate of return (before tax)	4.25%	4.25%	4.25%	4.25%
Income tax rate (inc. federal & state)	43.8%	43.8%	43.8%	43.8%
Hurdle rate of return (after tax)	2.39%	2.39%	2.4%	2.39%
Upfront grant or shared cost	$0	$0	$0	$0
Loan percentage of investment	100%	100%	100%	100%
Loan closing costs	4% of loan	4% of loan	4% of loan	4% of loan
Loan term	20 years	20 years	20 years	20 years
Loan (Bond) annual interest rate	4.25%	4.25%	**8.56%**	4.25%
Electricity Production Tax Credit (PTC) ($/kWh)	$0.022	$0.022	$0.022	**$0.000**
Power Purchase Agreement rate (PPA) ($/kWh)	$0.106	**$0.012**	$0.106	$0.106
Scenario Output				
Payback Period, non-discounted — before taxes & financing	9 years	13 years	9 years	9 years
Payback Period, non-discounted — after taxes & financing	N/A*	N/A*	N/A*	N/A*
Net Present Value (NPV) — before taxes & financing	$13,438,213	$4,540,128	$13,438,213	$13,438,213
Net Present Value (NPV) — after taxes & financing	$7,657,060	$0	$0	$7,588,426
Annualized NPV — before taxes & financing	$969,612	$327,585	$969,612	$969,612
Annualized NPV — after taxes & financing	$474,682	$0	$0	$470,427
Modified Internal Rate of Return (MIRR) — before taxes & financing	6.8%	5.3%	6.8%	6.8%
Modified Internal Rate of Return (MIRR) — after taxes & financing	N/A**	2.4%	2.4%	N/A**

*The project is 100% debt financed for 20 years **MIRR = N/A because the cash flow is positive in every year. This is a high MIRR.

Table 7: Simple Cash Flow for system using non-mean-reverting expected prices and not capturing full tax shield subsidies

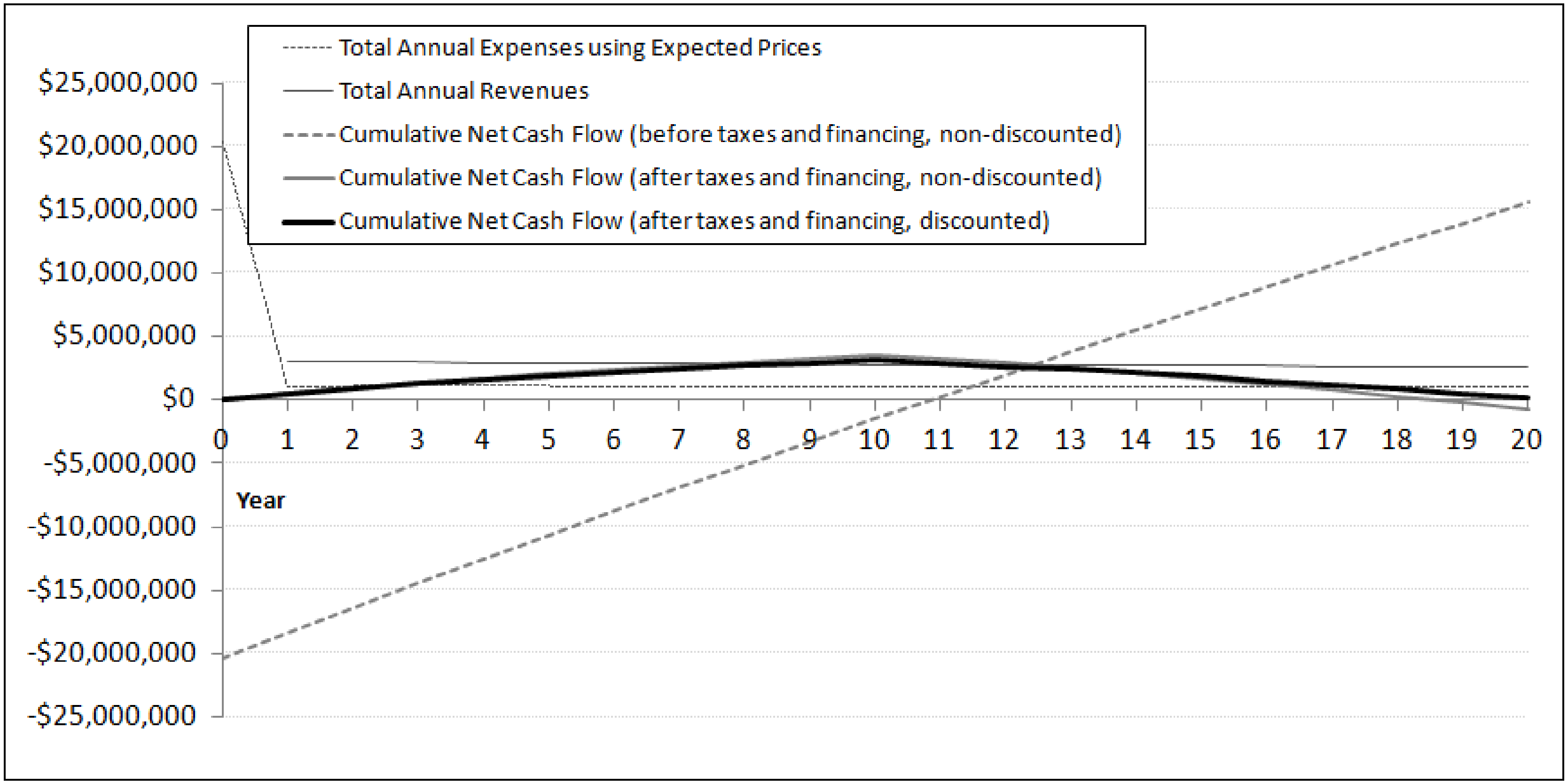


Figure 24: Cumulative Cash Flows of Non-Incremental Project using <u>mean-reverting</u> expected prices when full tax shield subsidies go unutilized

Scenario Assumptions		Baseline Scenario	Lower PPA Rate	Maximum Interest Rate	No PTC
Useful life		20 years	20 years	20 years	20 years
Years of depreciation		10 years	10 years	10 years	10 years
Hurdle rate of return (before tax)		4.25%	4.25%	4.25%	4.25%
Income tax rate (inc. federal & state)		43.8%	43.8%	43.8%	43.8%
Hurdle rate of return (after tax)		2.39%	2.39%	2.4%	2.39%
Upfront grant or shared cost		$0	$0	$0	$0
Loan percentage of investment		100%	100%	100%	100%
Loan closing costs		4% of loan	4% of loan	4% of loan	4% of loan
Loan term		20 years	20 years	20 years	20 years
Loan (Bond) annual interest rate		4.25%	4.25%	**4.34%**	4.25%
Electricity Production Tax Credit (PTC) ($/kWh)		$0.022	$0.022	$0.022	**$0.000**
Power Purchase Agreement rate (PPA) ($/kWh)		$0.106	**$0.104**	$0.106	$0.106
Scenario Output					
Payback Period, non-discounted	before taxes & financing	11 years	11 years	11 years	11 years
	after taxes & financing	N/A*	N/A*	N/A*	N/A*
Net Present Value (NPV)	before taxes & financing	$3,877,227	$3,702,915	$3,877,227	$3,877,227
	after taxes & financing	$166,646	$0	$0	$166,646
Annualized NPV	before taxes & financing	$279,755	$267,178	$279,755	$279,755
	after taxes & financing	$10,331	$0	$0	$10,331
Modified Internal Rate of Return (MIRR)	before taxes & financing	5.1%	5.1%	5.1%	5.1%
	after taxes & financing	2.7%	2.4%	2.4%	2.7%

*The project is 100% debt financed for 20 years

Table 8: Simple Cash Flow for complete system using mean-reverting expected prices and not capturing full tax shield subsidies

liabilities) enters into a contractual agreement with a partner who is seeking to reduce tax liabilities. Sharif, *et al.*, describe three common contractual arrangements: the "partnership flip," the "sale leaseback," and the "inverted lease." Tax equity partnerships have been particularly common in the wind energy sector.

In the Baseline Scenarios shown in Figures 23 and 24, as well as Tables 7 and 8, we displayed the investment metrics as they would be if the tax shield subsidies offered by the Federal Government in excess of the tax liabilities for the project itself went *unutilized*. This simple scenario, however, would mean foregoing $2,860,831 to $3,359,678 (depending on whether the mean-reverting or non-mean-reverting model is used) over the first five years in potential tax shield subsidies arising from the electricity Production Tax Credit (PTC) and accelerated capital depreciation (because the project does not generate enough profits in those years to generate a tax liability as high as the allowable deductions). Figures 25 and 26, as well as Tables 9 and 10 show the cumulative cash flow and the cash flow metrics, assuming the full value of the tax shield equity is accrued to the district heating project, using an arrangement such as a partnership flip, sale leaseback, or inverted lease (Sharif, et al., 11-15). The tax shield benefits would be shared by the parties in the arrangement, but we do not make assumptions about how the value of these tax subsidies would be divided between the parties. Since our focus is on the project itself, we treat the partners as if they were one entity and the full benefits of the tax equity accrue to the project.

In the non-mean-reverting version of the model, the value of the otherwise-unused tax subsidies (measured in terms of the increase in the after-tax NPV of the discounted

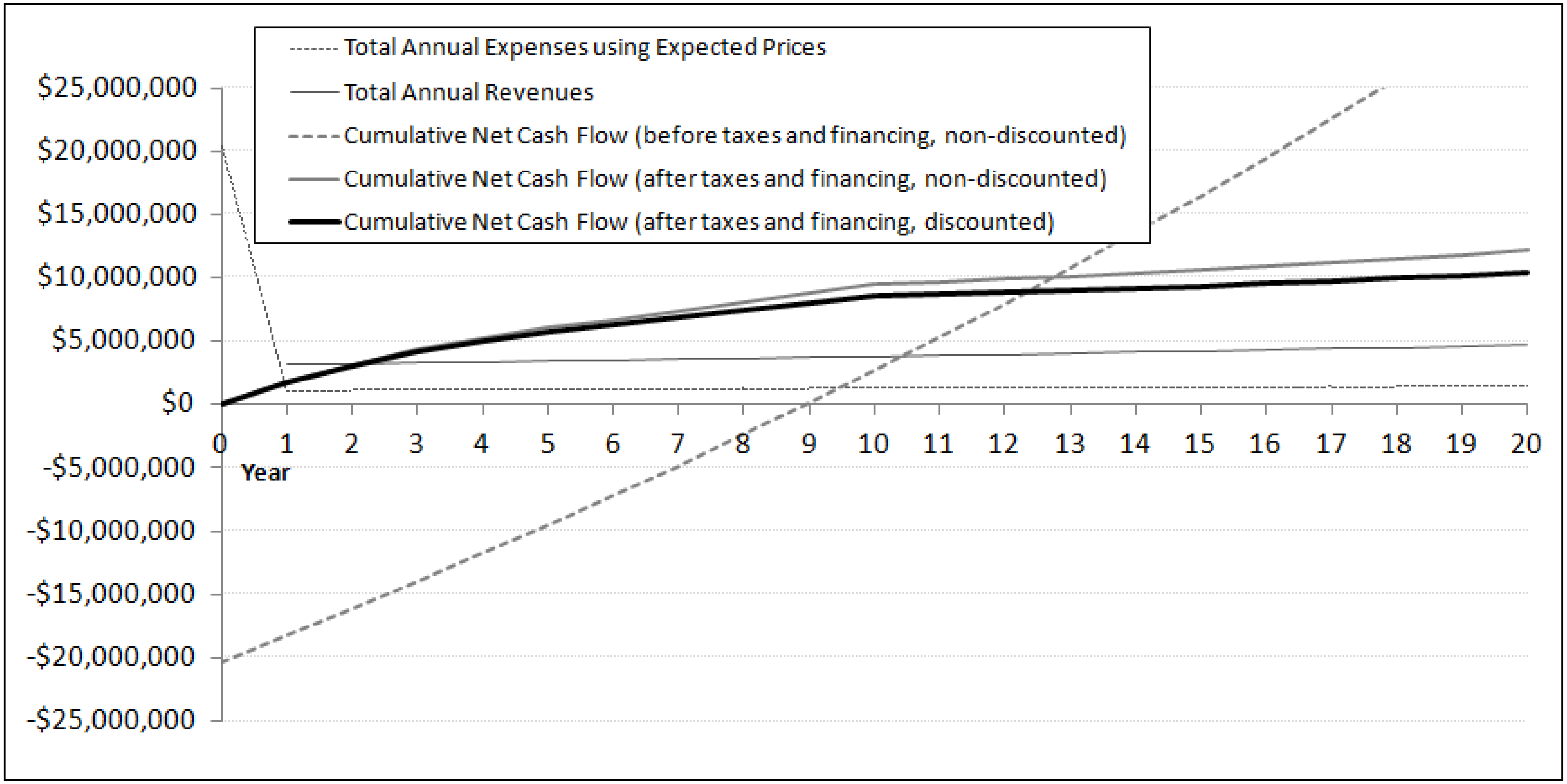


Figure 25: Cumulative Cash Flows of Non-Incremental Project using <u>non-mean reverting</u> expected prices when full tax shield subsidies are captured

Scenario Assumptions		Baseline Scenario	Lower PPA Rate	Maximum Interest Rate	No PTC
Useful life		20 years	20 years	20 years	20 years
Years of depreciation		10 years	10 years	10 years	10 years
Hurdle rate of return (before tax)		4.25%	4.25%	4.25%	4.25%
Income tax rate (inc. federal & state)		43.8%	43.8%	43.8%	43.8%
Hurdle rate of return (after tax)		2.39%	2.39%	2.4%	2.39%
Upfront grant or shared cost		$0	$0	$0	$0
Loan percentage of investment		100%	100%	100%	100%
Loan closing costs		4% of loan	4% of loan	4% of loan	4% of loan
Loan term		20 years	20 years	20 years	20 years
Loan (Bond) annual interest rate		4.25%	4.25%	**12.11%**	4.25%
Electricity Production Tax Credit (PTC) ($/kWh)		$0.022	$0.022	$0.022	**$0.000**
Power Purchase Agreement rate (PPA) ($/kWh)		$0.106	**-$0.056**	$0.106	$0.106
Scenario Output					
Payback Period, non-discounted	before taxes & financing	9 years	16 years	9 years	9 years
	after taxes & financing	N/A*	N/A*	N/A*	N/A*
Net Present Value (NPV)	before taxes & financing	$13,438,213	-$1,823,883	$13,438,213	$13,438,213
	after taxes & financing	$10,386,935	$0	$0	$10,241,071
Annualized NPV	before taxes & financing	$969,612	-$131,599	$969,612	$969,612
	after taxes & financing	$643,915	$0	$0	$634,872
Modified Internal Rate of Return (MIRR)	before taxes & financing	6.8%	3.8%	6.8%	6.8%
	after taxes & financing	N/A**	2.4%	2.4%	N/A**

*The project is 100% debt financed for 20 years **MIRR = N/A because the cash flow is positive in every year. This is a high MIRR.

Table 9: Simple Cash Flow for complete system using non-mean-reverting expected prices and full tax shield subsidies

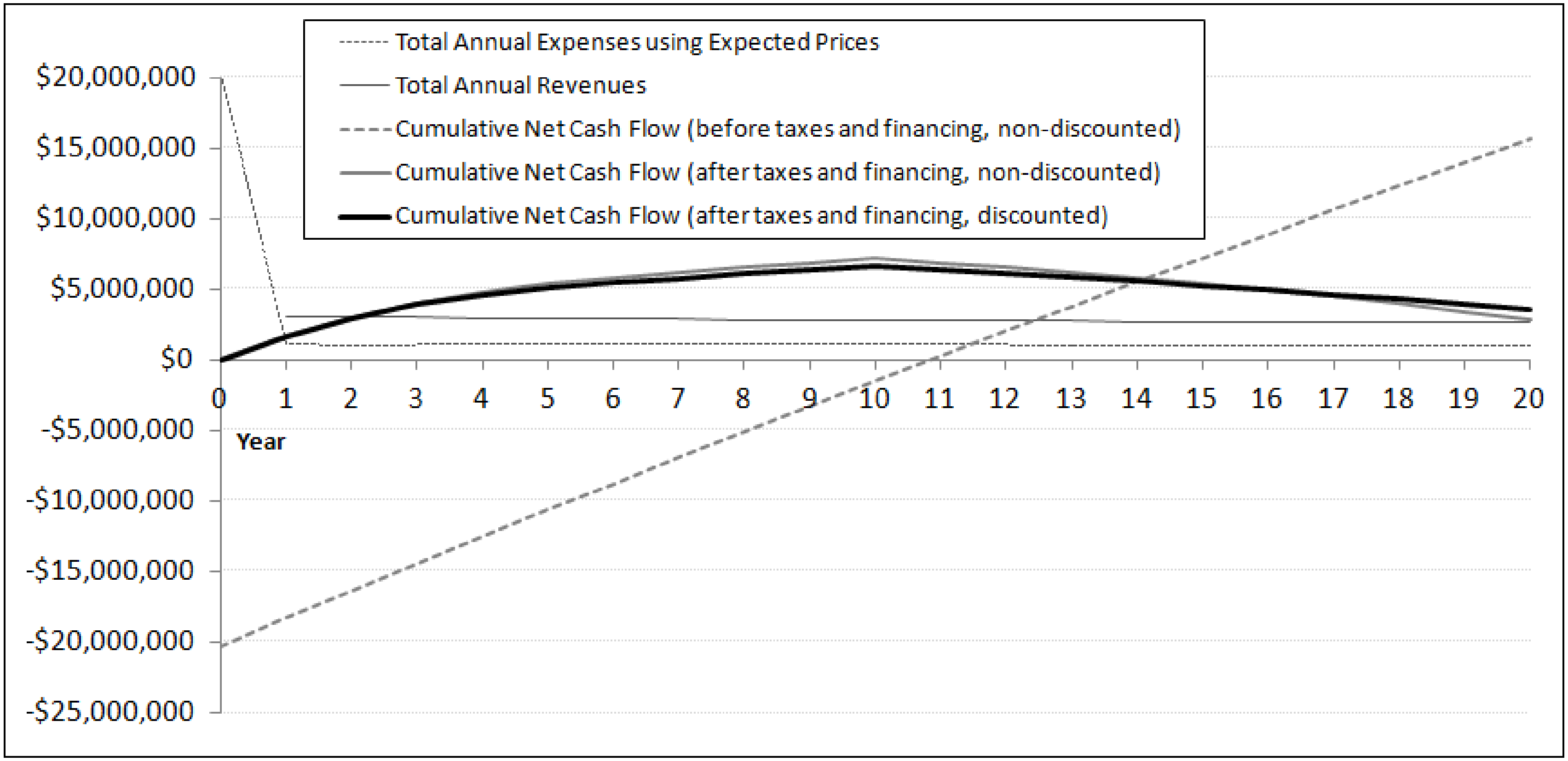


Figure 26: Cumulative Cash Flows of Non-Incremental Project using <u>mean reverting</u> expected prices when full tax shield subsidies are captured

Scenario Assumptions	Baseline Scenario	Lower PPA Rate	Maximum Interest Rate	No PTC
Useful life	20 years	20 years	20 years	20 years
Years of depreciation	10 years	10 years	10 years	10 years
Hurdle rate of return (before tax)	4.25%	4.25%	4.25%	4.25%
Income tax rate (inc. federal & state)	43.8%	43.8%	43.8%	43.8%
Hurdle rate of return (after tax)	2.39%	2.39%	2.4%	2.39%
Upfront grant or shared cost	$0	$0	$0	$0
Loan percentage of investment	100%	100%	100%	100%
Loan closing costs	4% of loan	4% of loan	4% of loan	4% of loan
Loan term	20 years	20 years	20 years	20 years
Loan (Bond) annual interest rate	4.25%	4.25%	**7.21%**	4.25%
Electricity Production Tax Credit (PTC) ($/kWh)	$0.022	$0.022	$0.022	**$0.000**
Power Purchase Agreement rate (PPA) ($/kWh)	$0.106	**$0.050**	$0.106	$0.106
Scenario Output				
Payback Period, non-discounted — before taxes & financing	11 years	15 years	11 years	11 years
Payback Period, non-discounted — after taxes & financing	N/A*	N/A*	N/A*	N/A*
Net Present Value (NPV) — before taxes & financing	$3,877,227	-$1,402,061	$3,877,227	$3,877,227
Net Present Value (NPV) — after taxes & financing	$3,592,929	$0	$0	$3,447,064
Annualized NPV — before taxes & financing	$279,755	-$101,163	$279,755	$279,755
Annualized NPV — after taxes & financing	$222,736	$0	$0	$213,693
Modified Internal Rate of Return (MIRR) — before taxes & financing	5.1%	3.9%	5.1%	5.1%
Modified Internal Rate of Return (MIRR) — after taxes & financing	6.3%	2.4%	2.4%	6.2%

*The project is 100% debt financed for 20 years

Table 10: Simple Cash Flow for complete system using mean-reverting expected prices and capturing full tax shield subsidies

project) is $2,729,875. In the mean-reverting version of the model, the value of the tax subsidies is $3,426,283.

G. The Basic Cash Flow Model Using Stochastic Price Distributions

In the next stage of the analysis we add additional sophistication to the basic cash flow model by making use of the distributions that we calculated around the input and output prices in Section 3(D). Using a risk analysis add-in software for Microsoft Excel called @Risk[14] it is possible to easily run a Monte Carlo analysis within the model, in which the prices in each time period are randomly drawn from the distributions that we defined using either the Dual-Factor Heteroskedstic Stochastic Diffusion technique (for the mean-reverting price model) or the GBM technique (for the non-mean-reverting price model). By quickly running 10,000 independent simulations (in less than five minutes), we can develop a very granular distribution of possible outcomes for each of our performance metrics.

In the non-mean-reverting version of the model, when the full value of the tax subsidies *is not* captured, the NPV (after taxes) is positive in 61% of the 10,000 simulations, but the distribution has a long tail in the negative direction, reflecting the fact that the scenarios of great unprofitability are more extreme than the scenarios of great profitability. If the full value of the tax subsidies were captured, the NPV (after taxes) would be positive in 73% of the 10,000 simulations (see Figure 27).

In the mean-reverting version of the model, when the full value of the tax subsidies *is not* captured, the NPV (after taxes) is positive in 30% of the 10,000 simulations. If the

[14] A product of the Palisade Corporation

full value of the tax subsidies *were* captured, the NPV (after taxes) would be positive in 97% of the simulations.

When using the mean-reverting model, the large increase in the number of simulations in which the NPV would be positive is partly related to the lower standard deviation of the distributions in the mean-reverting model. This is an interesting point that will become important when we look at Real Options analysis. The standard deviations of the NPV's without mean reversion in Figure 27 are approximately $26,000,000. Looking at Figure 28, where the mean-reverting model is used, the standard deviations of the NPV's are only about $1,600,000. This is because the assumption of mean-reversion helps to increase the certainty around future prices, and reduce the number of extremely high or low price scenarios. It also means that when the distribution is shifted a certain amount to the right (as it is when tax subsidies are fully captured), a larger portion of the distribution crosses into the positive quadrant on the charts using the mean-reverting model than on the charts using the non-mean-reverting GBM model.

We have seen that under an ideal scenario, where the full value of tax subsidies are captured by the investor and the investor receives a $0.106/kWh (electric) PPA rate for the electricity produced by the system, the after-tax NPV for the system is highly positive ($3,592,929; this is the "Baseline Scenario" column in Table 10). Furthermore, Figure 28 shows that the NPV for this scenario is positive over 97% of the time. However, several factors could be of concern to an investor. As we noted previously, there is no guarantee that the investor will receive a PPA rate of $0.106/kWh (electric). Butcher noted that a rate half of that would be more reflective of the avoided market cost of electricity (arguably). Cutting the PPA rate in half would put the project precariously close to *un*profitability (see the second column on Table 10, with the lower PPA rate). It also remains a concern that after taxes and financing, the annual discounted Net Cash Flow is consistently below zero after year 10, when the PTC and accelerated depreciation expire (again, see Figure 26).

These factors might make an investor more wary of the project. So referring back to the decision diagram in Figure 2, up to this point the analysis has shown that given most assumptions, Alternative 1 would undesirable, or there are factors that would be a cause of concern for an investor. The investor might therefore be interested in Alternative 2 (only building the District Heating system, if it would be profitable by itself) or Alternative 3 (only building the District Heating system because of the value of the option to phase in CHP). In order to analyze either of these alternatives, we need to break the investment into two parts. This leads us to the next element of the model, which is an exploration of Real Options. In this case, we are exploring the option of first installing only biomass-powered District Heating equipment (no electricity production), and then deciding later whether to expand this equipment to offer biomass-powered CHP (in which the residual heat from the electricity production is used in the District Heating grid).

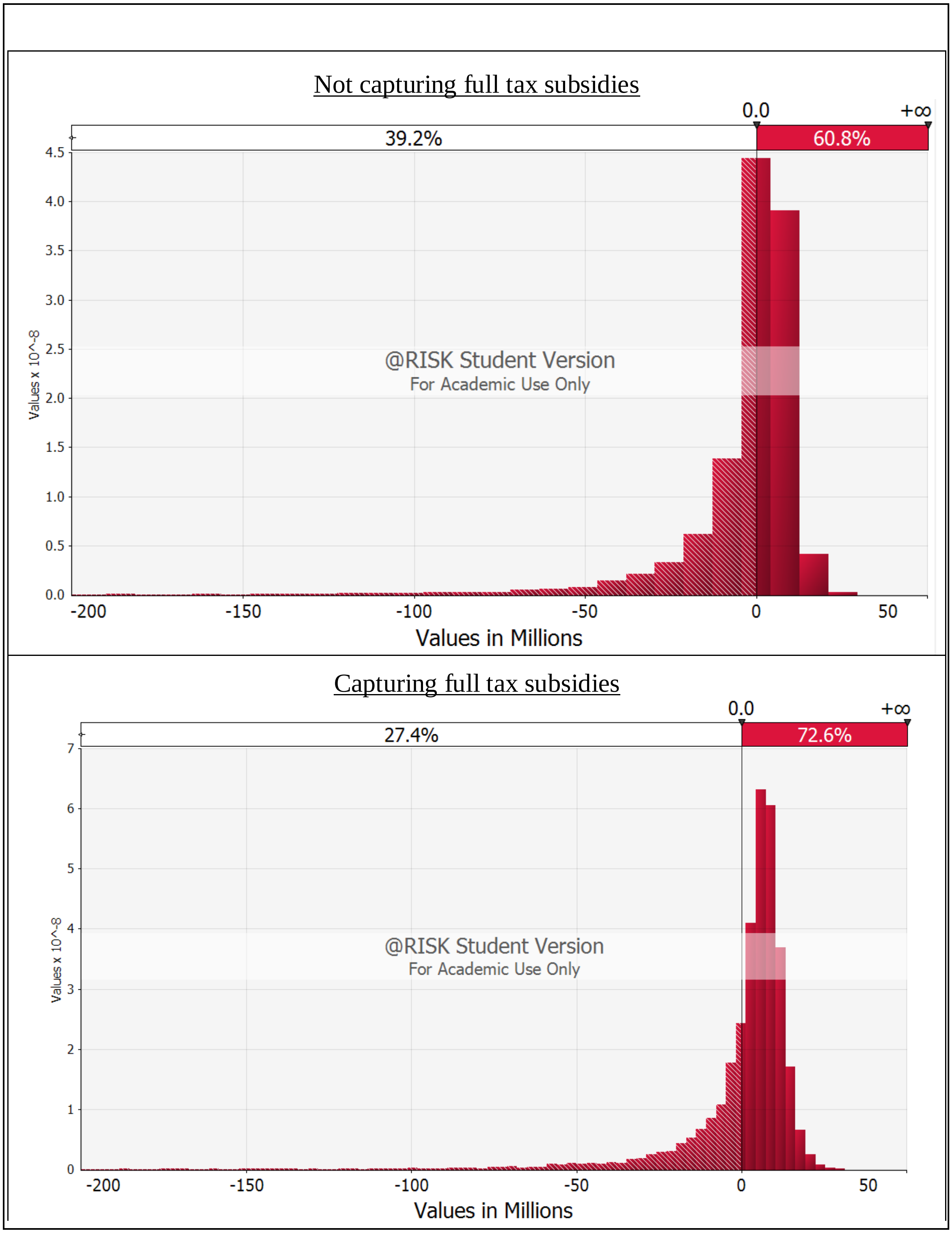


Figure 27: Stochastic distribution of the after-tax NPV for complete system using non-mean-reverting prices

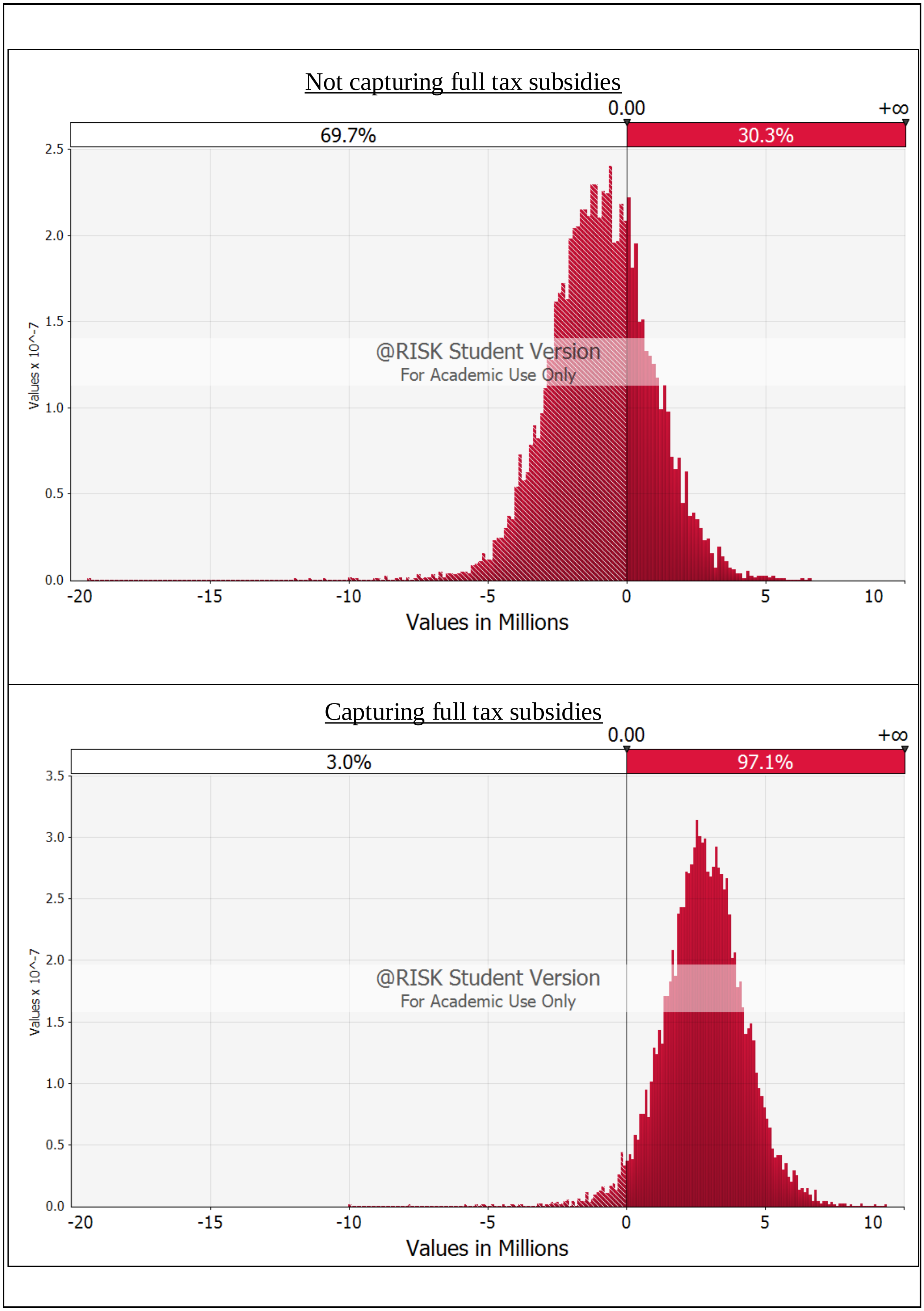


Figure 28: Stochastic distribution of the after-tax NPV for complete system using mean-reverting prices

4. REAL OPTIONS AND THE ABILITY TO EXPAND INTO CHP

Financial options, such as those that are bought and sold for agricultural commodities, are purchased contracts that give the purchaser the right (but not the obligation) to buy or sell some asset for a certain price at the date of the option's expiration (or often at any point *until* the contract's expiration) (Ross, Westerfield and Jordan 430-431). They allow investors and hedgers flexibility in managing market risk when buying or selling stocks, bonds, or commodity contracts. In comparison, a "*real* option" enables an investor to manage market risk, but involves a real capital investment (Ross, Westerfield and Jordan 448). This is often a matter of investment timing. An investor may perceive that the market could potentially shift, making the investment profitable at a later date. Examples of such a shift would be a change in commodity prices or renewable energy production incentives.

Specifically, we are here discussing a managerial option to expand, since the District Heating/CHP project could potentially be broken into its two component parts (District Heating and then CHP). An NPV analysis that only considers the *complete* project may underestimate the value of the initial District Heating investment, since this initial investment would be able to generate a cash flow on its own and is a necessary initial step toward installing a combined DH/CHP facility – a step that may be desirable *depending on how prices resolve themselves.* Similarly, a NPV analysis of only the initial DH project would ignore the potentially valuable option to expand into CHP that the DH project would provide to the owner.

In their pioneering 1973 paper titled "The Pricing of Options and Corporate Liabilities," authors Fischer Black and Myron Scholes derived a formula to attribute a theoretical value to these so-called "European options," which allow their holders to exercise them on specified future date. The Black-Scholes model continues to be very widely-used in modern finance. It is also utilized for Real Options valuation, where the option to expand can be modeled as if it were a financial call option. Importantly, even though the Black-Scholes model applies specifically to European options, the value of a European call option is the same as the value of a so-called "American call option," which allows its holder to exercise it <u>at any point up until</u> some specified future date (Benninga 435 Chapter 16 §6 Proposition 2). (Note, however, that the value of a European *put* option would not be the same as the value of an American put option.) We can therefore use the Black-Scholes model when valuing the option to expand from DH to CHP as if it were a financial call option.

The Black-Scholes model follows the following formulas (shown here as interpreted by Benninga) (*Ibid.* 509 Chapter 19 §2):

$$C = SN(d_1) - Xe^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where the value of the call = C, the price of the underlying stock = S, the exercise price of the call = X, the call's time to exercise = T, the interest rate = r, and the standard deviation of the logarithm of the stock's return = $\sigma. N()$ represents the value of the standard normal distribution.

Interpreting these values from the perspective of Real Options, C = the NPV of the option to expand, S = the NPV of additional cash flows generated by the expansion, X = the cost of the expansion, T = the time until the expansion is made, r = the risk-free interest rate, and σ = the percent volatility of the expansion's expected returns.

A. The Initial DH Investment With and Without Mean-Reverting Prices

In order to solve for the necessary inputs to the Black-Scholes options pricing model, we must separate the project into its two phases. The capital costs are already tracked separately in Table 3. We can see that by separating the project into two phases, the investors would limit their initial risk to the $10,497,596 that is required to invest in the Phase 1 heat-only subset of the overall project. The investor would retain the option of investing in the second half of the project (the CHP portion) at the cost of an additional $9,056,441. By once again inputting the costs into an annual cash flow model using the same scenario assumptions shown in Table 7 and Table 8, we can look at the performance metrics for the first phase of the project as a stand-alone piece, both with and without mean reversion. For simplification, in the Real Options analysis, we always assume that the full value of tax subsidies are captured. The results for the non-mean-reverting model are shown in Figure 29 and Table 11. The results from the mean-reverting model are shown in Figure 31 and Table 12. As before, we present the performance metrics for Phase 1 of the project in a simple manner using expected prices in Figure 29 and Figure 30, and then we present the stochastic distributions of the NPV in in Figure 31. Looking at the distributions in Figure 31, we can again see that the assumption of mean reversion narrows the distribution around the potential NPV outcomes for Phase 1 of the project by itself, but it also reduces the expected return of the project. This is the same effect that was observed when we used mean-reverting prices to model the complete system.

When using the mean-reverting price model the Phase 1 District Heating system alone never satisfies the profitability requirement that the after-tax NPV be greater than zero. If using expected prices, as shown in Figure 31 and Table 12, the after-tax NPV equals -$2,987,202. And in the stochastic analysis, we can see from the lower distribution in Figure 31 that the number of scenarios generating a positive NPV is essentially 0%. If using non-mean-reverting prices in the model, the expected prices do, however, result in a positive expected after-tax NPV of $3,648,261. Here again, however, it is useful to examine the stochastic output in the upper distribution in Figure 31. The 10,000 simulations show a much longer tail in the unprofitable direction. This occurs particularly in situations where the value of the heat output drops within the first five years (a scenario that is undeniably possible). It is difficult for the investment valuation to recover from an early drop partly because future income streams are discounted. The shape of this NPV distribution indicates that this is a risky investment, despite the fact that expected future prices show a very positive after-tax NPV. This is a situation in which a stochastic analysis looking at thousands of different iterations using software such as

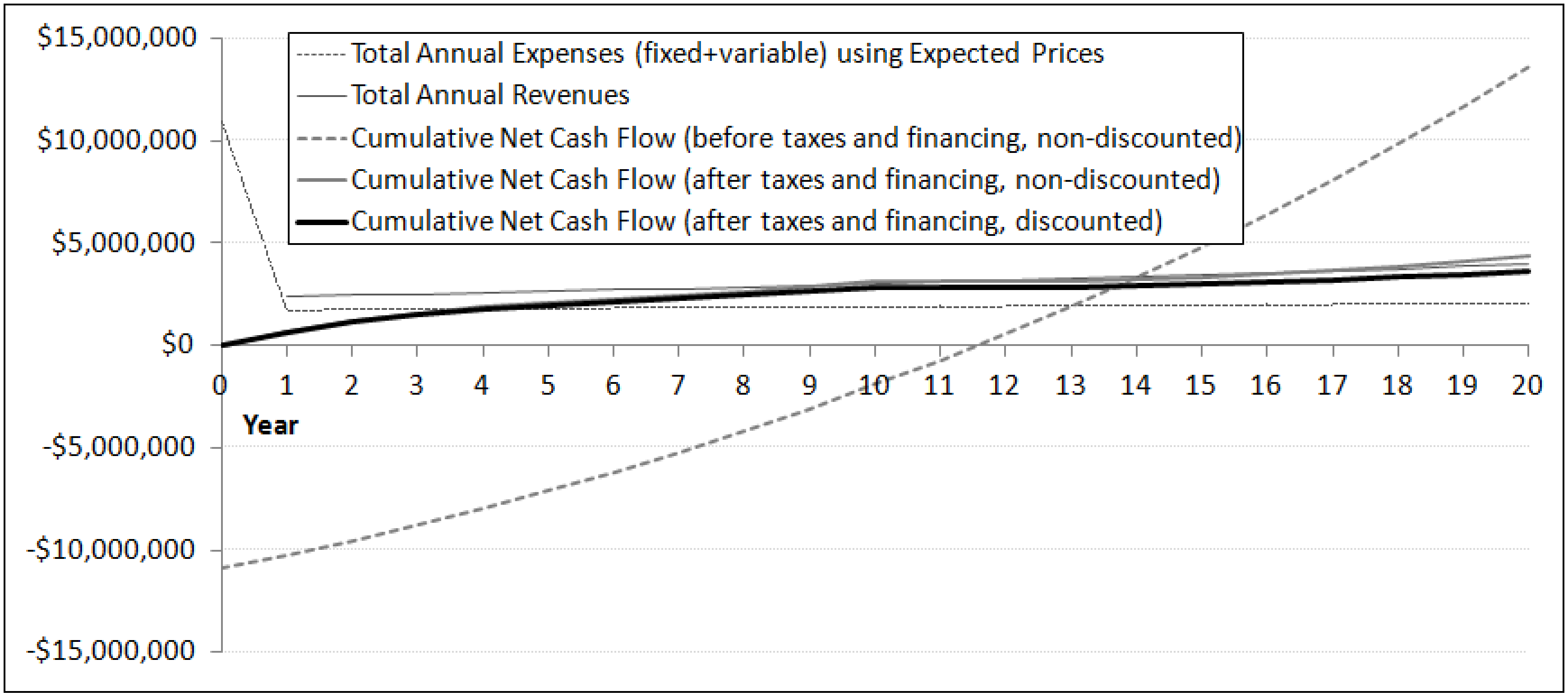


Figure 29: PHASE 1 Cumulative Cash Flows using non-mean reverting expected prices when full tax shield subsidies are utilized

Scenario Assumptions		Baseline Scenario	Varying the starting Wood Pellet Price	Maximum Interest Rate
Useful life		20 years	20 years	20 years
Years of depreciation		10 years	10 years	10 years
Hurdle rate of return (before tax)		4.25%	4.25%	4.25%
Income tax rate (inc. federal & state)		43.8%	43.8%	43.8%
Hurdle rate of return (after tax)		2.39%	2.39%	2.39%
Upfront grant or shared cost		$0	$0	$0
Loan percentage of investment		100%	100%	100%
Loan closing costs		4% of loan	4% of loan	4% of loan
Loan term		20 years	20 years	20 years
Loan (Bond) annual interest rate		4.25%	4.25%	**9.61%**
Electricity Production Tax Credit (PTC) ($/kWh)		N/A	N/A	N/A
Starting Wood Pellet Price (Year 0, $/ton)		$230.00	**$286.48**	$230.00
Power Purchase Agreement rate (PPA) ($/kWh)		N/A	N/A	N/A
Scenario Output				
Payback Period, non-discounted	before taxes & financing	12 years	17 years	12 years
	after taxes & financing	N/A*	N/A*	N/A*
Net Present Value (NPV)	before taxes & financing	$4,204,895	-$1,241,722	$4,204,895
	after taxes & financing	$3,648,261	$0	$0
Annualized NPV	before taxes & financing	$303,397	-$89,594	$303,397
	after taxes & financing	$226,166	$0	$0
Modified Internal Rate of Return (MIRR)	before taxes & financing	5.9%	3.7%	5.9%
	after taxes & financing	37.9%	2.4%	2.4%

*The project is 100% debt financed for 20 years

Table 11: PHASE 1 Simple Cash Flow Metrics of the system using <u>non-mean-reverting</u> expected prices and tax subsidies

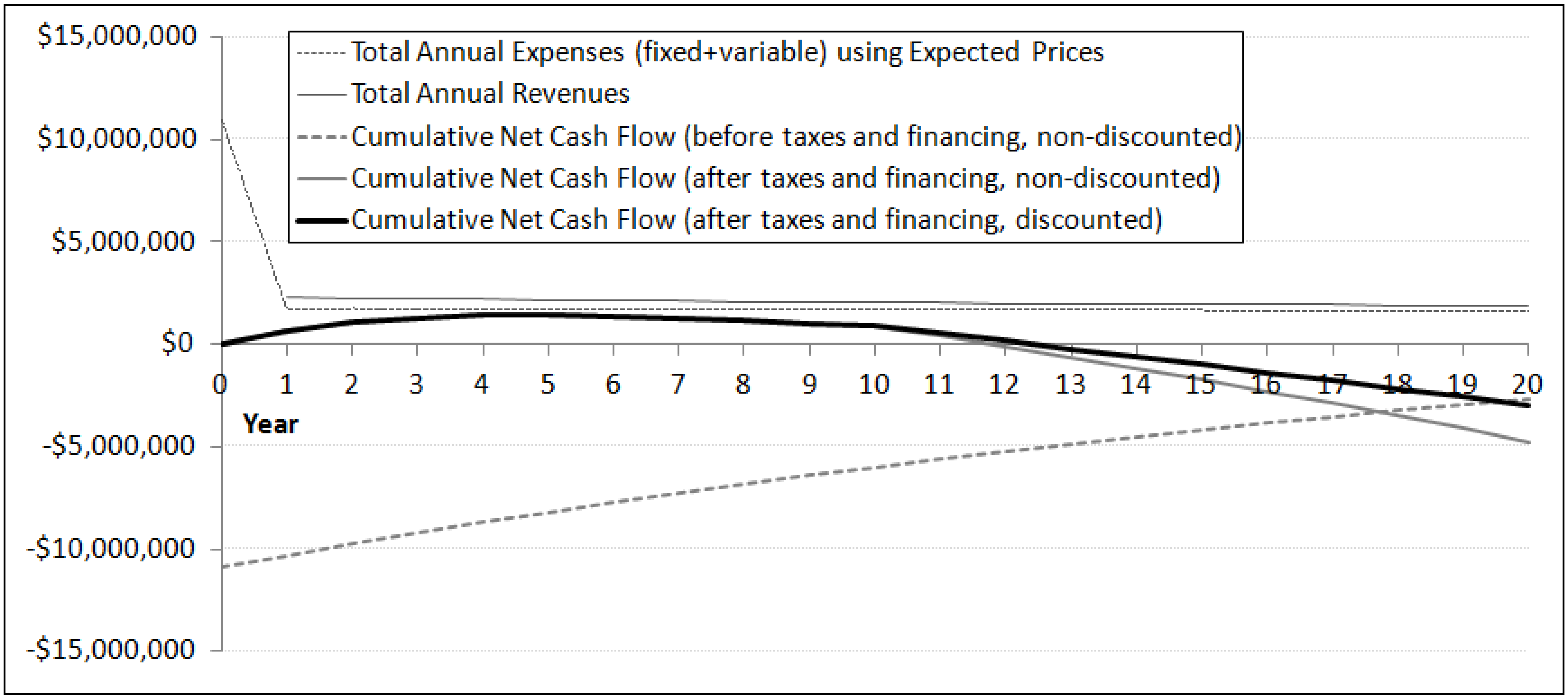


Figure 30: PHASE 1 Cumulative Cash Flows using <u>mean reverting</u> expected prices when full tax shield subsidies are utilized

Scenario Assumptions		Baseline Scenario	Varying the starting Wood Pellet Price	With Grant Funding
Useful life		20 years	20 years	20 years
Years of depreciation		10 years	10 years	10 years
Hurdle rate of return (before tax)		4.25%	4.25%	4.25%
Income tax rate (inc. federal & state)		43.8%	43.8%	43.8%
Hurdle rate of return (after tax)		2.39%	2.39%	2.39%
Upfront grant or shared cost		$0	$0	$3,129,783
Loan percentage of investment		100%	100%	100%
Loan closing costs		4% of loan	4% of loan	4% of loan
Loan term		20 years	20 years	20 years
Loan (Bond) annual interest rate		4.25%	4.25%	4.25%
Electricity Production Tax Credit (PTC) ($/kWh)		N/A	N/A	N/A
Starting Wood Pellet Price (Year 0, $/ton)		$230.00	$176.76	$230.00
Power Purchase Agreement rate (PPA) ($/kWh)		N/A	N/A	N/A
Scenario Output				
Payback Period, non-discounted	before taxes & financing	>20 years	14 years	18 years
	after taxes & financing	N/A*	N/A*	N/A*
Net Present Value (NPV)	before taxes & financing	-$5,127,491	-$631,383	-$1,872,517
	after taxes & financing	-$2,987,202	$0	$0
Annualized NPV	before taxes & financing	-$369,966	-$45,556	-$135,108
	after taxes & financing	-$185,185	$0	$0
Modified Internal Rate of Return (MIRR)	before taxes & financing	1.2%	4.0%	2.9%
	after taxes & financing	Negative	2.4%	2.4%

*The project is 100% debt financed for 20 years.

Table 12: PHASE 1 Simple Cash Flow Metrics of the system using mean-reverting expected prices and full tax subsidies

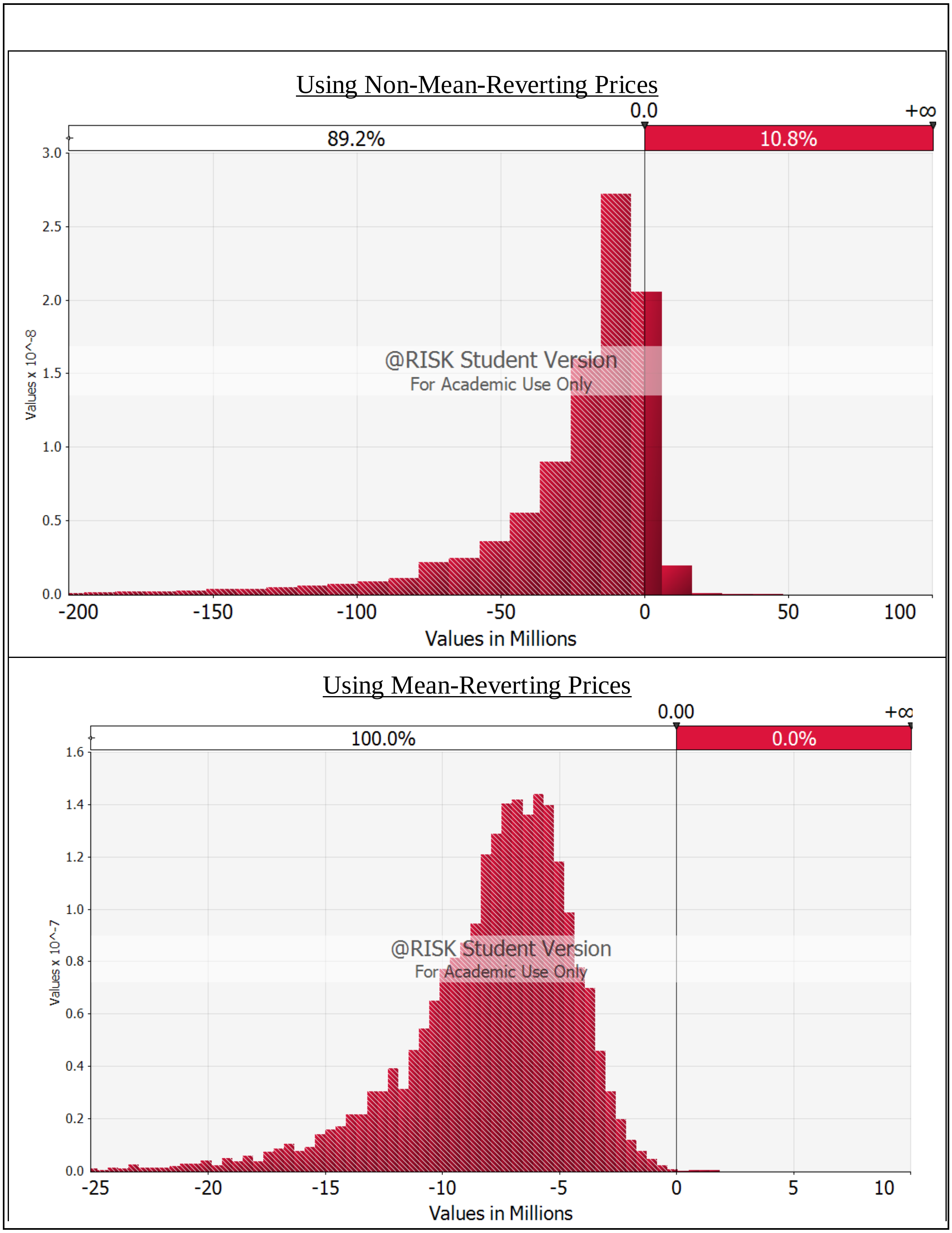


Figure 31: Stochastic distribution of the after-tax NPV for Phase 1 of the system using the non-mean-reverting and mean-reverting price models

@Risk would be very useful to a potential investor. The disproportionately risky distribution of negative possible outcomes (given the input and output price distributions) means that the mean of the 10,000 possible after-tax NPV values is quite low. It is -$27,091,845 in the non-mean-revering price model and-$7,950,967 in the mean-reverting model.

So investors who assume mean-reverting energy prices would most certainly not be attracted by the District Heating project by its own merits alone. Investors who assume *non*-mean-reverting expected energy prices might find the District Heating project alone to be attractive – until they examine the distribution of possible returns, which shows that the project is only profitable in 10.8% of the cases. Referring back to the decision possibilities shown in Figure 2, this would mean that Alternative 2 is undesirable in most scenarios.

Part of the disadvantage of the Phase 1 District Heating system by itself is that its operating costs would be higher than if the system were operating in tandem with the CHP system: by itself, the District Heating system would need to use wood pellet boilers, because its energy demand is not large enough to cope with the less consistent TBB fuel that would substantially lower fuel costs. In Table 11 and Table 12 we present a starting wood pellet price that would be necessary to make the Phase 1 project feasible by itself alone. The price of $176.76/ton in Table 12 is roughly three-quarters of the current $230.00/ton is currently found in this part of Minnesota.

Another disadvantage for the Phase 1 District Heating project alone is that there are fewer production subsidies available for renewable **heat** generation (unlike the incentives that the U.S. Federal and State governments offer for renewable electricity generation). The primary subsidies offered in the United States for renewable heat generation come in the form of project grants from the Department of Energy and the Department of Agriculture. In the mean-reverting price model the project would require a grant of $3,129,783 to make the Phase 1 District Heating project profitable by itself alone (but this is a non-stochastic calculation; remember that the potential for downside risk would still be greater than the upside potential).

On the other hand, it is possible that a government grant would not necessarily need to be this large in order to entice investor(s) to the project. The initial Phase 1 DH investment carries with it a potentially valuable option to expand into CHP. This option to expand potentially carries with it a value that could lower the enticement that would be necessary in the form of a grant subsidy for Phase 1 of the project. In the following sections we calculate and analyze the value of that option. Essentially, the following analysis will show us whether an investor in this project could end up with Alternative 3 or Alternative 4 from Figure 2.

B. The Option to Expand on the Initial DH Investment with CHP

While in most instances the NPV of the initial DH investment is negative, the NPV alone could be misleading since it ignores the fact that the DH investment is a necessary first step toward (possibly) expanding into CHP. Estimating the value of the option to expand using the Black-Scholes options pricing model requires us to bring together everything we have developed up to this point.

First, we use the *difference* in the cash flows between the initial investment and the complete CHP system to solve for the value of S in the Black-Scholes equation (equal to the Present Value (PV) of the cash flows generated by the complete system minus the cash flows generated by the initial DH investment).[15] X = the cost of the expansion, T = the time until the expansion is made, and r = the risk-free interest rate. We use the @Risk software to model σ = the percent volatility of the expansion's expected returns. The mean reversion process that we solved for earlier makes it possible to plug into the annualized Excel cash flow model a discrete, yearly value for the standard deviation of the uncertain input prices. The @Risk software then runs thousands of random scenarios and provides a very granular distribution for the final, differenced PV, after factoring in the uncertainty of the investment's input and output prices. The standard deviation of this distribution divided by the mean of the expected additional cash flows equals the percent volatility in the project's expected returns, or σ in the Black-Scholes options pricing model. Finally, we use the Black-Scholes model to solve for C = the value of the option

[15] Note that here the Present Value is used, and not the *Net* Present Value. This is because the cost of the initial investment enters into the Black-Scholes equation via the variable X.
Also note that because the project is 100% financed, when calculating S we differenced the Present Values of the cash flows *before* deducting the principal repayments. This was necessary, because if we lowered the cash flows by the amount of the principal repayments, we would decrease S in the Black-Scholes equation, but we would also decrease X (the cost of exercising the option). By financing the project at 100% S would decline if we deducted principal repayments, but X would effectively decrease to zero, meaning there is no cost to exercising the option. This would make the Black-Scholes equation irrational. For the sake of calculating S in the Black-Scholes equation, therefore, we use the Present Value of the additional cash flows *not deducting* principal repayments, and then for X in the Black-Scholes equation we use the full cost of the expansion, not reduced by the amount of any loans. We still deduct interest payments from the cash flows, because they are costs to the project that are not already included in the value of X.

to expand. If the value of C compensates for the negative NPV of the initial DH project, then the value of the option to expand is worth the expected loss of the initial investment – just like it would be for financial hedgers who purchase call options that they may or may not exercise because the options provide the opportunity, but not the obligation, to make a future purchase (Amram & Kulatilaka 122).

We have chosen to use five years as the value for T, representing the time before the expansion would be made (that is, the time during which the investor would decide whether or not to embark upon the CHP expansion). This assumption could be varied. If it were higher, then the value of the option would increase.

Table 13 shows the option output that we use as a baseline, assuming either the mean-reverting model and the non-mean-reverting model. Using the inputs under current assumptions, the project now satisfies the conditions that we have set for deeming the investment profitable. That is, using the non-mean-reverting price model, the option value of $33,060,067 compensates for the average negative NPV of the simulations from the initial investment, which was in the range of -$26,700,000. (Note that the exact figure for the NPV of the initial project may change slightly when we re-run multiple scenarios during the sensitivity analysis. This is due to the stochastic elements in the price cash flows. The more iterations we run in each scenario, however, the closer we come to a consistent “actual” mean expected outcome.) In the mean-reverting price model, the Real-Options method of analysis also results in a profitable investment, but to a lesser extent. Using mean-reverting assumptions, the option value is $10,016,799, which compensates for the negative value of the initial investment, which was in the range of -$7,900,000.

Note again the way the reduced uncertainty in future prices in the mean-reverting price model influences both the mean value of the initial investment and also the value of the option to expand. As we observed before, the mean NPV of the initial investment is less negative using mean-reverting prices, because the potential of extremely unprofitable scenarios is reduced. However, the value of the Black-Scholes option price is also greatly reduced because the additional price information in the mean-reverting model helps narrow uncertainty. The much higher option value in the non-mean-reverting model demonstrates the point made by Laughton and Jacoby that we mentioned in Section 3(C), where they demonstrate that ignoring mean reversion can result in biased project valuations. Using the non-mean-reverting Real Options model, an investor would come up with a very convincingly high NPV for the project of $6,342,825 – this despite the fact that the profitability of the initial investment is quite uncertain and had an average stochastic NPV of -$26,717,242. (Or, to put a finer point on it, the Real Options model returns such a high value *exactly because* future prices are so uncertain and therefore result in an exceedingly high option volatility value of 179.06%.) For this reason, the much higher Real Options NPV attained using the non-mean-reverting model could be somewhat controvertible.

It is also worthwhile to note Table 14, in which we show the Real Options output under a scenario with mean-reverting prices but without capturing the full value of tax shield subsidies. This is not the scenario we have chosen for our “baseline,” because we

OPTION SUMMARY (NON-MEAN-REVERTING)	
Risk-free interest rate (r)	2.4%
Present Value of Additional Cash Flows (S)	$33,771,332
Option Exercise Price (X)	$9,056,441
Time until Option is Exercised (T) (years)	5
Option Volatility	179.06%
OPTION VALUE = BS CALL PRICE	$33,060,067
NPV OF INITIAL INVESTMENT	-$26,717,242
NPV OF TOTAL PROJECT (INITIAL INVESTMENT + OPTION TO EXPAND)	**$6,342,825**

OPTION SUMMARY (MEAN-REVERTING)	
Risk-free interest rate (r)	2.4%
Present Value of Additional Cash Flows (S)	$17,914,497
Option Exercise Price (X)	$9,056,441
Time until Option is Exercised (T) (years)	5
Option Volatility	22.50%
OPTION VALUE = BS CALL PRICE	$10,016,799
NPV OF INITIAL INVESTMENT	-$7,920,485
NPV OF TOTAL PROJECT (INITIAL INVESTMENT + OPTION TO EXPAND)	**$2,096,314**

Table 13: The initial DH project's value under the baseline conditions, after considering the option to expand into CHP, and capturing the full tax shield value of incentives

OPTION SUMMARY (EXC. TAX SHIELD VALUE)	
Risk-free interest rate (r)	2.4%
Present Value of Additional Cash Flows (S)	$18,432,821
Option Exercise Price (X)	$9,056,441
Time until Option is Exercised (T) (years)	5
Option Volatility	24.42%
OPTION VALUE = BS CALL PRICE	$10,577,012
NPV OF INITIAL INVESTMENT	-$12,174,574
NPV OF TOTAL PROJECT (INITIAL INVESTMENT + OPTION TO EXPAND)	**-$1,597,563**

Table 14: The initial DH project's value, after considering the option to expand into CHP; with mean-reverting prices, but without capturing the full value of tax shield subsidies

believe it would be more likely that the project would only be undertaken by an investor or group of investors who would be able to fully utilize this tax shield subsidies. However, it is interesting to note that without those subsidies, the project would *not* be considered profitable, even after including the value of the option to expand into CHP.

Note that when tax subsidies are not fully captured, the NPV of the initial DH investment declines by over $4 million in comparison to the mean-reverting scenario in Table 13, while the option value increases slightly. The NPV of the initial DH investment is lower without the full tax shield benefits because a large part of the additional tax shield comes from the tax shield value of depreciation (as well as the tax shield value of interest). This works out to be relatively more of a benefit in the first phase of the project, where annual revenues (and profits) are far lower, and where an investor with no other tax liabilities would not be paying nearly enough tax on the profits from the DH project to capture the full value of the depreciation. The ability to capture that tax shield value is therefore relatively more impactful in the first phase of the project. The relatively larger decline in NPV of the initial project has the additional impact of *increasing the difference* between the cash flows from the first phase and the second phases of the project, which is used in the Black-Scholes model as S. This results in a higher option value. Additionally, while the expected NPV for the second phase of the project declines when the tax shield benefits are not captured, the volatility of those cash flows (σ) increases slightly. A higher volatility means less certainty (which is bad for an investor), but a higher σ also means that the option's value is higher.

The net overall effect of capturing the full value of the tax shield benefits is to make the initial investment in District Heating appear rational from a Real Options perspective. Revisiting Figure 2, we can see that this is a non-negligible conclusion, because it means that Alternative 3 could present a rational path forward for District Heating projects, even though Alternatives 1 and 2 were shown to be undesirable in most cases when analyzed using stochastic cash flow methods. In order for Alternative 3 to present a realistic option forward, however, the investor must retain the option of expanding into CHP after 5 years (or more), and must capture the full tax-shield value of the DH project's depreciation (presumably by using it to lower other existing tax liabilities). Furthermore, Alternative 3 represents a rational path forward whether or not mean-reverting future prices are used, even though it could be argued that the Real Options value of the project could be more realistic using the mean-reverting price model, rather than the non-mean-reverting price model.

C. Real Options Sensitivity Analysis

Next we test the mean-reverting model using a series of alternate scenarios to see how the desirability of the project changes with different assumptions about key output prices and volatility. As our starting point we use the scenario shown in Table 13, in which the full tax shield value of the project's benefits are captured by the investor and prices are mean-reverting. The key output prices in the model are still those of heat and electricity, and the prices for the TBB and Pellet inputs are dependent upon the prices of heat and electricity (as we described earlier in Section 3). First we explore the impact of alternate future expected price scenarios (but retain the same expected volatilities around those prices). Second we retain the same expected future prices, but vary the expected volatilities around those prices. Finally, we explore the boundaries of possible scenarios by combining changes in expected prices with changes in prices volatilities.

1. Alternate Future Price Scenarios

The expected price progressions used in the initial model (using a Mean-Reverting Process) were shown in Figures 18 and 20. We provide three alternative scenarios: *Alt. Scenario 1* in which the annual change in expected prices is in the same direction as projected, but at double the rate (that is, * 2); *Alt. Scenario 2* in which the expected prices do not change – they are constant (* 0); and *Alt. Scenario 3* in which the expected prices change at the same rate as we expect, but in the opposite direction (* -1). Figures 32 and 33 display the expected prices over 20 years under these scenarios. Figure 34 shows the NPV output from running these scenarios. Note that in the original price progression scenarios, both the prices for heat and electricity declined over the next 20 years (this is an outcome of the mean-reverting price progression). In Figure 34, therefore, an x-value of *positive* 2.00 means that the prices *decline* twice as fast annually as originally projected in the mean-reverting baseline scenario. A value of *negative* 1 means that the prices *increase* at the same rate at which the model originally predicted they would decrease. Also note that by assuming the expected price changes by a constant percent per year, the NPV of the initial investment changes essentially linearly.

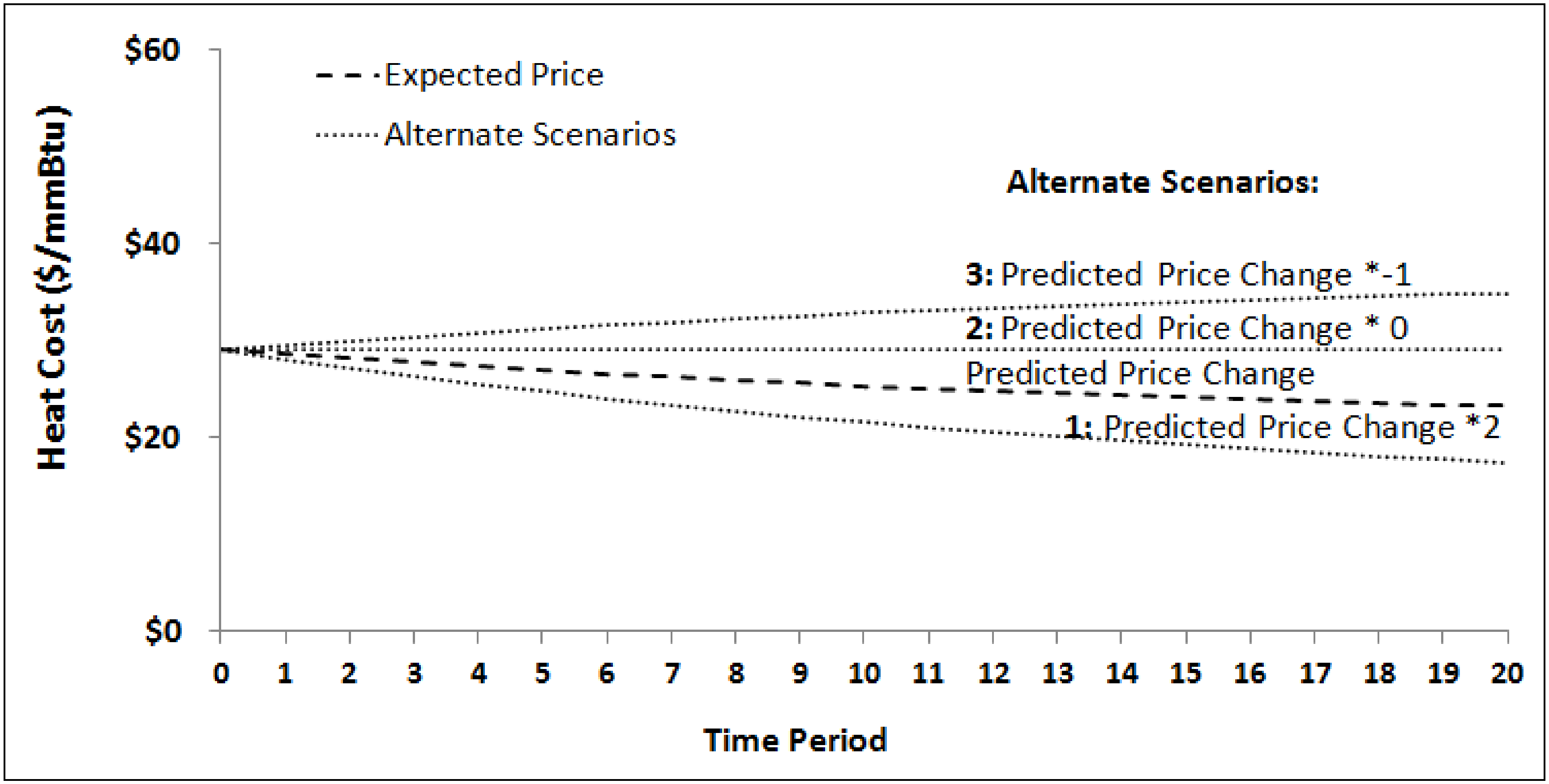


Figure 32: Expected Future Prices of Heat in Alternate Scenarios using a mean-reverting process

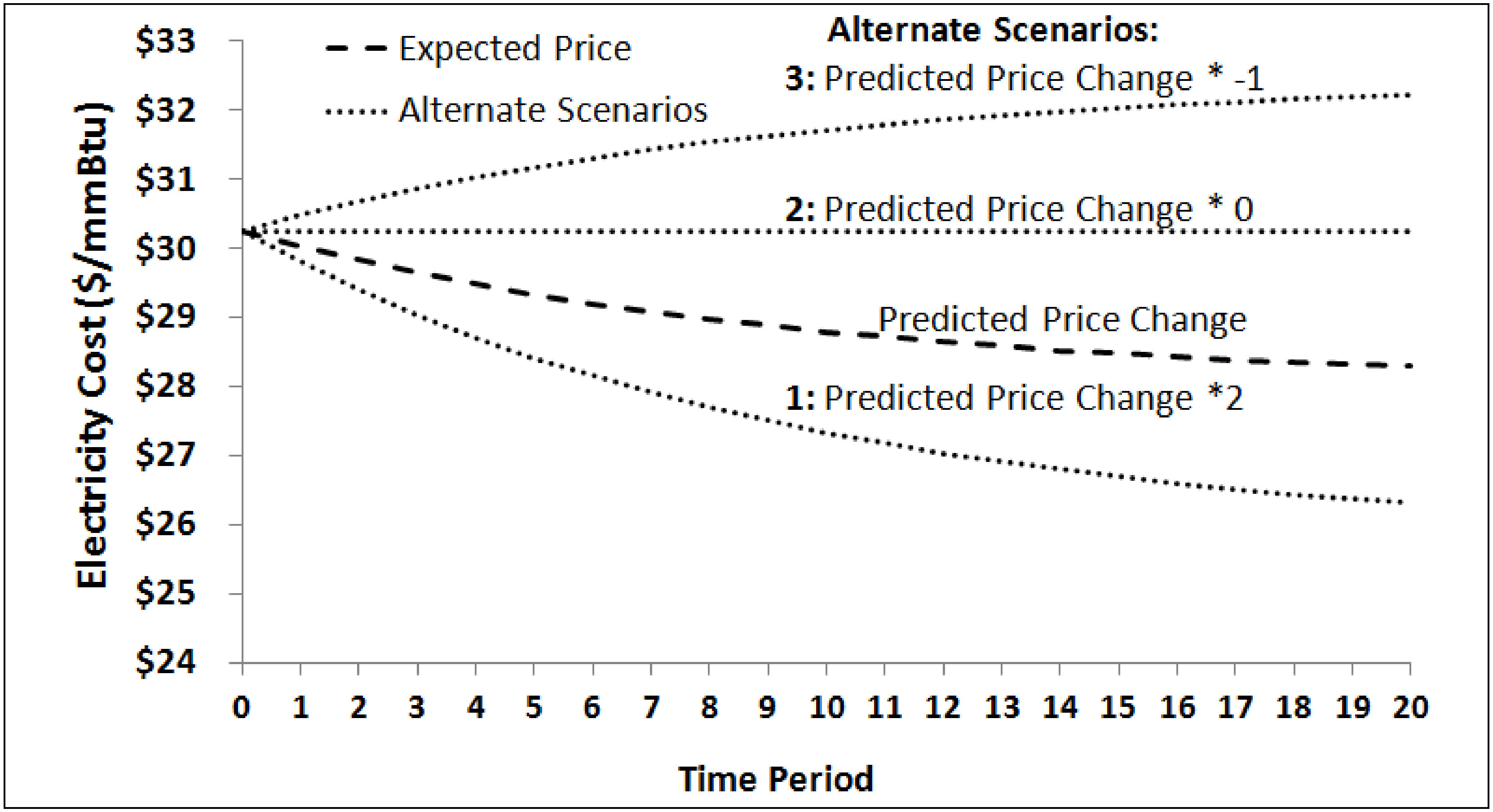


Figure 33: Expected Future Prices of Electricity in Alternate Scenarios using a mean-reverting process

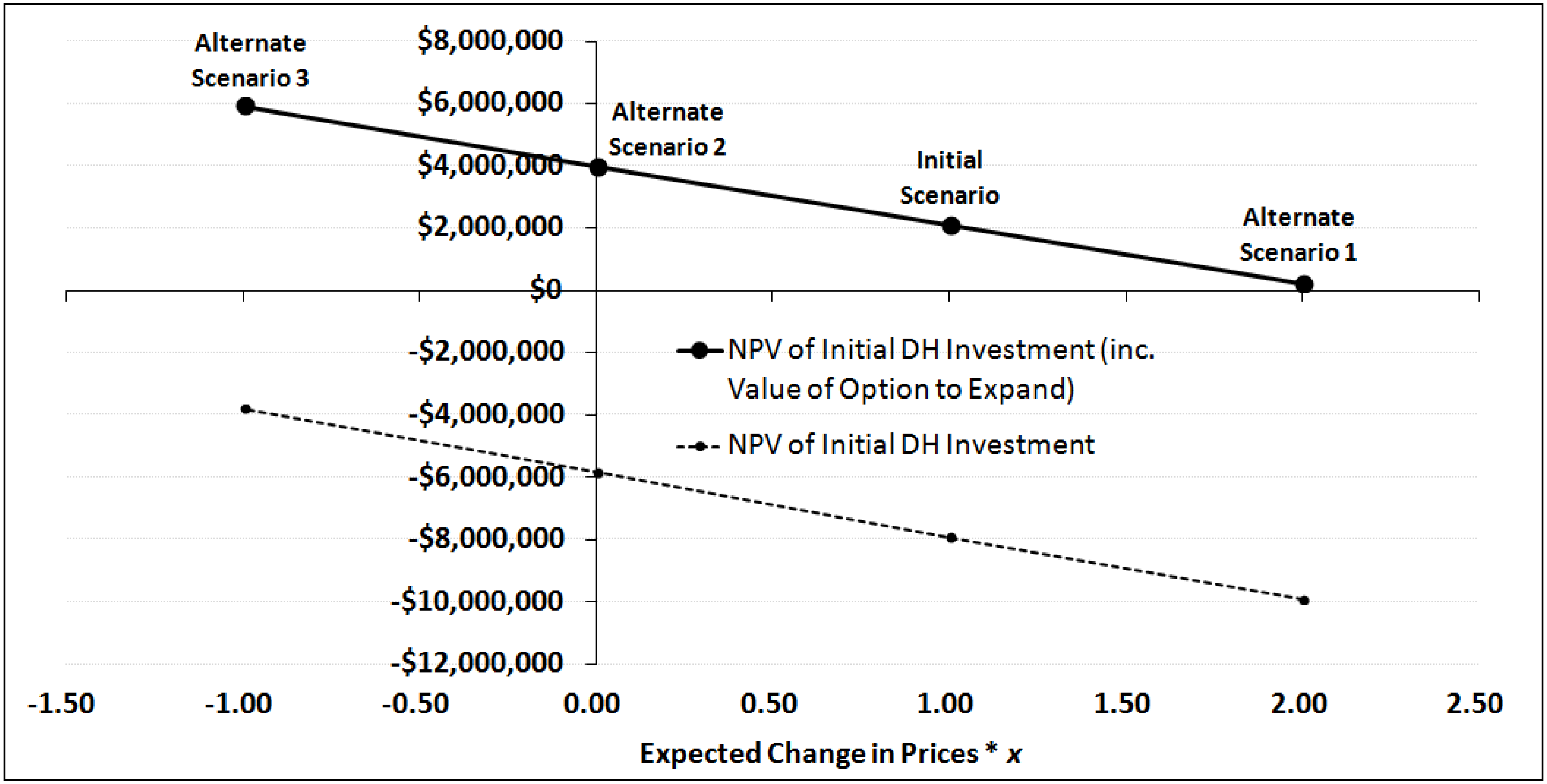


Figure 34: Real Options outcomes under alternate expected price scenarios

At $x = 1$ (the initial, baseline price scenario), the value of the option to expand just barely makes the overall investment positive. Changing the expected price progression results in a very minor change in the value of the option to expand. The impact is larger on the NPV of the initial DH investment. If heat and electricity prices decline more drastically than expected (as in $x = 2.00$), the value of the option to expand no longer compensates for the highly negative NPV of the initial DH investment, and the project would be undesirable from the perspective of a Real Options analysis.

2. Alternate Future Volatility Scenarios

In the next phase of sensitivity analysis, we examine impact of other volatilities around the expected price (as opposed to the volatilities shown in Figures 18 and 20). We examine four alternative scenarios: *Alt. Scenario 1* in which the volatility of both inputs is 1.50 times the expected baseline volatility; *Alt. Scenario 2* in which the volatility of both inputs is 1.25 times the expected baseline volatility; *Alt. Scenario 3* in which the volatility of both inputs is 0.75 times the expected baseline volatility; and *Alt. Scenario 4* in which the volatility of both inputs is 0.50 times the expected baseline volatility. On the input side, these scenarios are depicted visually in Figures 35 and 36.

The output from running these scenarios is shown in Figure 37. The net effect of varying the volatility assumptions is minimal, because higher price volatility increases the value of the option, but decreases the value of the underlying investment. The reason the value of the underlying investment decreases under higher volatility scenarios is because the upside potential of the DH project is more limited than the downside risk – a fact that we pointed out in Section 4.A when discussing the stochastic results of the DH investment alone. But as we have also already mentioned, the increase in volatility increases the value of the option. So the net effect of increasing the volatility is low, but the value of the project is still higher when volatility is lower.

3. Varying Both Expected Prices and Volatility in the Mean-Reverting Model

Finally, we conduct a two-dimension stochastic examination in the mean-reverting model by varying both expected future price progressions (as shown in Figures 32 and 33) and at the same time the volatilities around those prices (as shown in Figures 35 and 36). The results are given in Table 15. The initial mean-reverting "baseline" prices and volatilities are shown in **bold** lettering. The NPV of the initial DH investment is given in the bottom of the cell in *italicized* lettering (it is negative in all cases, meaning it would not be an attractive investment). The NPV including the option to expand from DH into CHP is <u>underlined</u>. Scenarios in which this value is positive have a black background (meaning the value of the option to expand into CHP makes up for the negative NPV of the initial DH investment). Scenarios in which this value is negative have a white background (meaning the project is never attractive). Changes to expected future prices have a larger impact on profitability than do changes to the volatility of future prices.

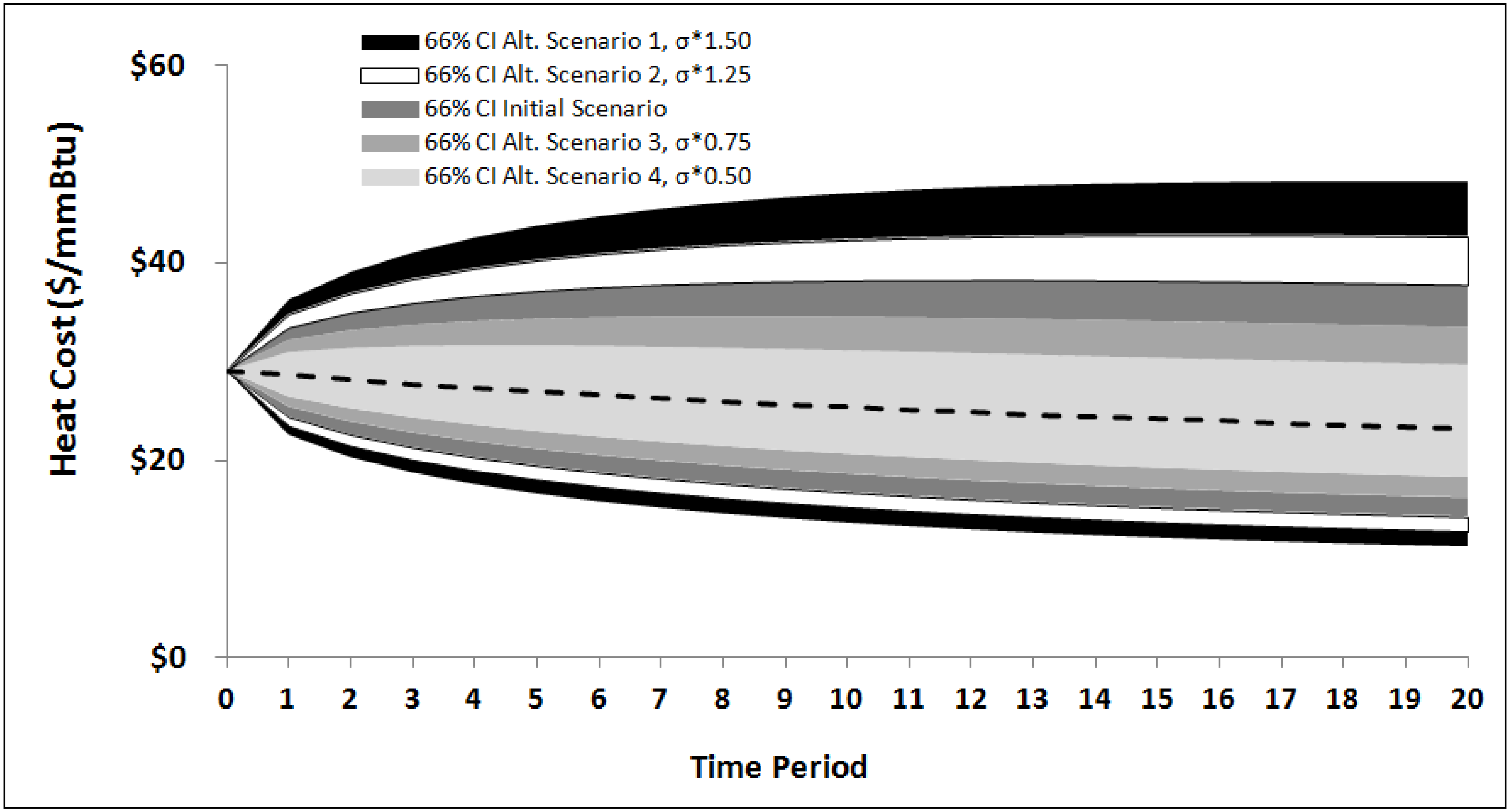


Figure 35: Expected future volatility of heat prices in alternate scenarios using a mean-reverting process

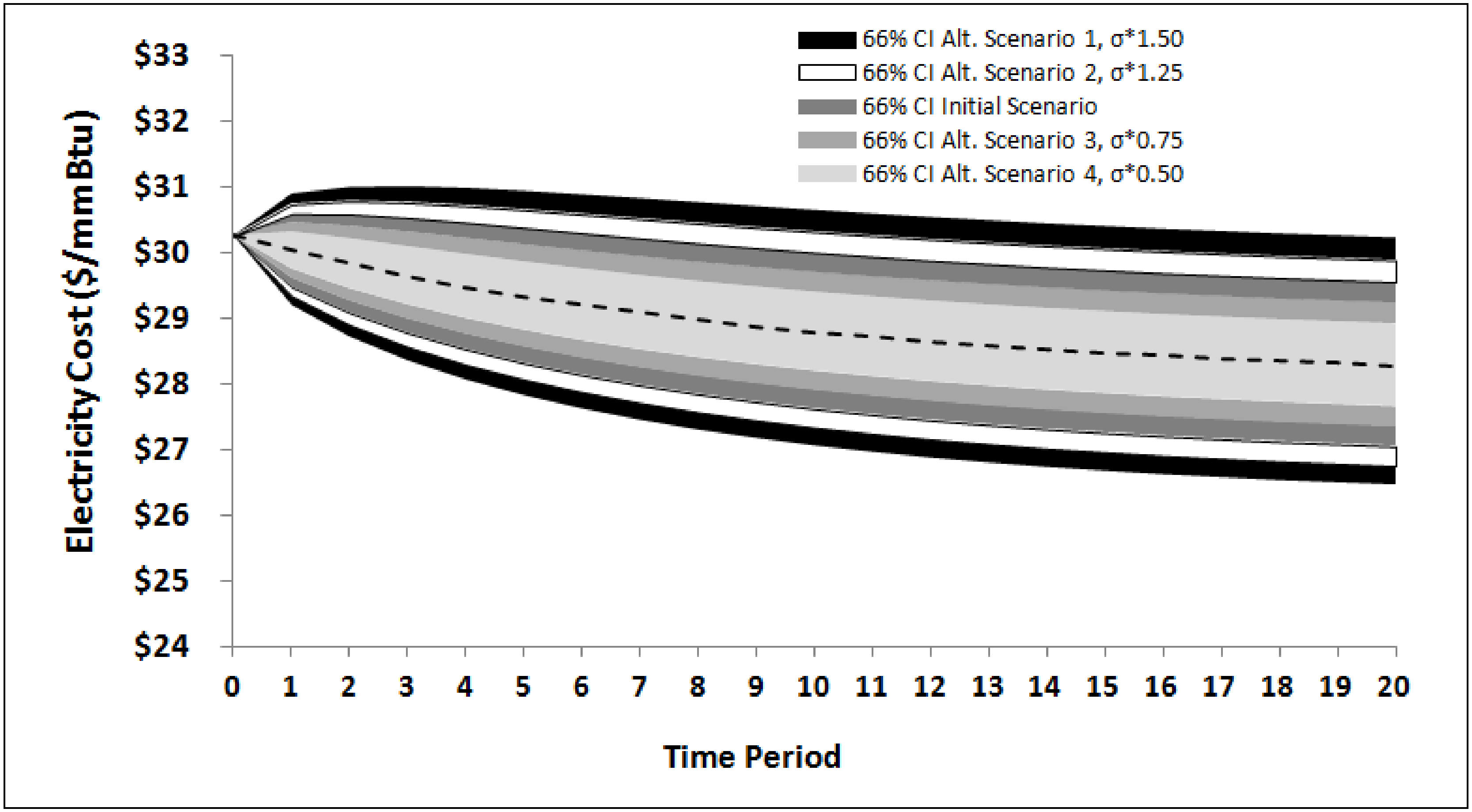


Figure 36: Expected future volatility of electricity prices in alternate scenarios using a mean-reverting process

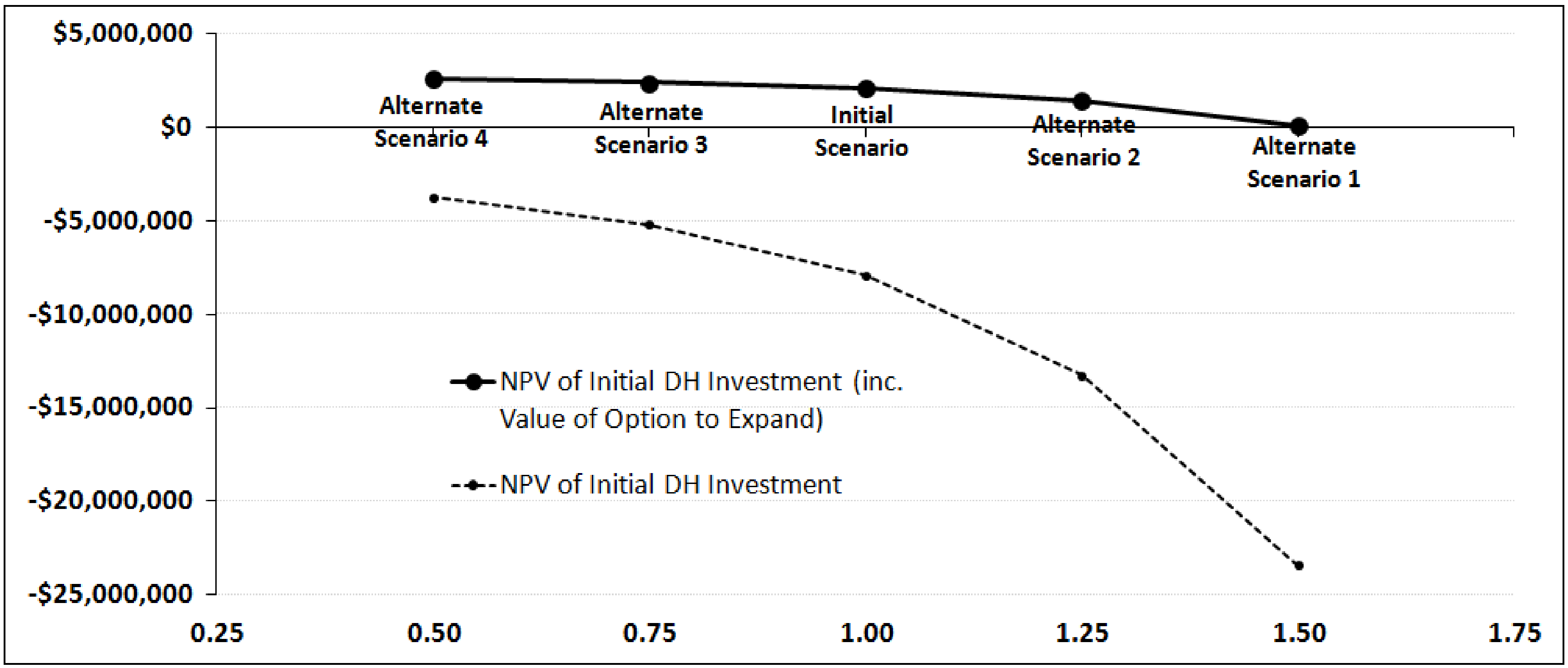


Figure 37: Real Options outcomes under alternate price volatility scenarios

NPV SENSITIVITY		**Expected price σ * *x***				
		0.50	**0.75**	**1.00**	**1.25**	**1.50**
NPV... Inc. option to expand *initial DH investment*	**expected annual change in log(price) * *x*** **2**	$776,231 *-$5,728,435*	$573,058 *-$7,250,673*	**$207,402** ***-$9,929,435***	-$428,041 *-14,948,941*	-$1,663,049 *-24,270,941*
	1	**$2,620,256** ***-$3,729,632***	**$2,411,091** ***-$5,173,234***	**$2,096,314** ***-$7,920,485***	**$1,450,629** ***-$13,259,179***	**$138,854** ***-$23,425,542***
	0	$4,503,844 *-$1,841,911*	$4,301,671 *-$3,129,161*	**$3,967,264** ***-$5,837,840***	$3,487,806 *-$11,265,431*	$2,201,480 *-$21,905,154*
	-1	$6,405,385 *$15,836*	$6,215,720 *-$1,153,293*	**$5,910,190** ***-$3,800,793***	$5,540,342 *-$9,403,376*	$4,238,425 *-$20,718,103*

Table 15: Results of a Two-Factor stochastic examination of project feasibility including the value of the option to expand from DH into CHP

(Scenario results using initial mean-reverting "baseline" prices and volatilities are shown in **bold** lettering.
The NPV of the initial DH investment is given in the bottom of the cell in *italicized* lettering.
The NPV including the option to expand from DH into CHP is underlined.
Scenarios in which the overall NPV is positive have a black background.
Scenarios in which the overall NPV is negative have a white background.)

5. CONCLUSIONS

When evaluating the desirability of a potential project, it is frequently necessary for an investor to make assumptions about future prices and about alternative phases or scales of the investment. It is not always realistic to simply continue the current price trends indefinitely into the future, particularly when the project's useful life extends out several decades. Similarly, it is unrealistic to ignore the value that a business might place on the opportunity to learn from an initial, smaller investment that would open the door to expand into a larger project.

In this paper we undertook an analysis of the future energy prices that determine the economic feasibility of a biomass-powered district heating and CHP system for a small city in northern Minnesota. We developed our own VBA code to solve for future prices using a binomial mean-reverting model. We found that heat and electricity prices are currently at a high level due to a recent upward trend that the mean-reverting model shows is unlikely to continue. We found that a longer-term (mean-reverting) tendency of these expected prices would predict a general price decline in real terms over the lifetime of this district energy project. We also found that a mean-reverting analysis greatly narrows the band of uncertainty around possible future prices.

Using these two different price models we then developed a cash flow model of the complete district energy system (district heating plus CHP) in Microsoft Excel. This cash flow model showed that if non-mean-reverting prices are used, the complete district energy project usually passes the test of financial feasibility. However, the cash flow model also showed very high that if mean-reverting prices are used, the complete district energy project usually fails the test of financial feasibility (the mean average after-tax NPV of all scenarios is negative, even though in the scenario with expected prices it is positive).

Next, using Real Options analysis techniques, we split the district energy system into two incremental pieces: first, a biomass-powered district heating grid, and then an electricity and heat-producing CHP system. We showed that a possible path forward for district energy systems is this incremental approach. By starting with district heating and then waiting to decide whether or not to expand into CHP, there is enough value in the *option* to expand that it makes the otherwise-unattractive district heating investment a possibility for a profit-seeking investor expecting a 4.25% annual rate of return.

In order to find the edge of profitability and understand the potential risk, we repeated the mean-reverting analysis multiple times using stochastic prices and varying the fundamental assumptions about future prices and volatility. We found that higher levels of price volatility and faster rates of price declines made the project less profitable. Conversely, lower price volatility and slower rates of price declines made it more profitable.

In conclusion, this analysis demonstrates that a financial analysis incorporating mean-reverting commodity prices represents a more "conservative" method of analysis when compared with Geometric Brownian Motion commodity price forecasts, because it results in fewer scenarios under which the project would be considered profitable. However, splitting the project into two pieces and using Real Options analysis techniques shows that even when using the more conservative mean-reverting commodity prices, there is potential value in the initial District Heating investment, because of the option that it potentially provides to more easily expand into Combine Heat and Power at some point in the future.

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7. APPENDICES

A. Appendix A: Lognormal Distribution of Energy Prices

It is often assumed that energy prices are lognormally distributed, because they can never fall below zero. To represent overall energy prices, we use daily New York Harbor Conventional Regular gasoline prices from 1986 to the present , and then we translate the historical prices to present-day dollars using the quarterly Gross Domestic Product index from the Department of Commerce. The distribution of this data is generally lognormal, as can be seen from the graph in Figure 38 that was generated by @Risk software. In fact, the Lognormal distribution is one of the better-fitting distributions according to its Chi-squared statistic (see Table 16).

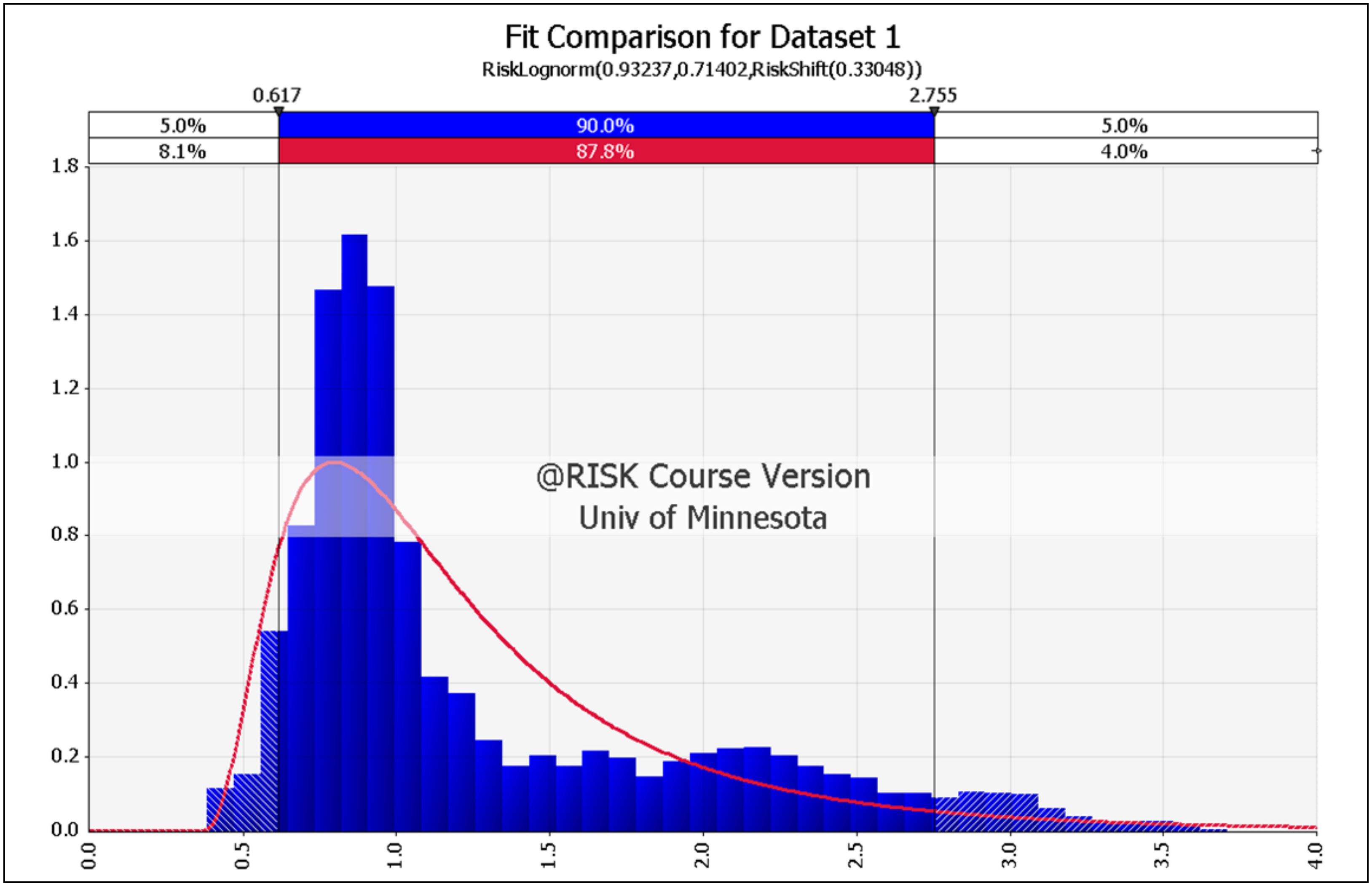

Figure 38: Distribution of New York Harbor Conventional Regular gasoline prices from 1986 to Present, showing lognormal distribution

Chi-sq Statistic for distribution fitting	
InvGauss	1617.53
Pearson5	1620.76
Lognorm	1672.29
LogLogistic	1844.80
ExtValue	3057.68

Table 16: Chi-Squared Statistics for various distribution fittings for New York Harbor Conventional Regular gasoline prices from 1986 to Present

B. Appendix B: VBA Code to solve for Two-Product Mean Reversion Parameters and Annual Prices and Volatilities

```
Sub Mean_Reversion()


Dim StkX(), StkY(), StkXpa(), StkYpa(), Opt(), StkXY(), Euro(), A(), B(), C(), D(),
vX(), vY(), pu, pd, puu(), pud(), pdu(), pdd(), puIFu, pdIFu, puIFd, pdIFd, CostX(),
CuuXtemp(), CudXtemp(), CduXtemp(), CddXtemp(), ChangeX(), Count(), Totalprob(),
Timeprob(), Movmnt(), Expectedmovmnt(), ExpPuutemp(), ExpPudtemp(),
ExpPdutemp(), ExpPddtemp(), ExpPuu(), ExpPud(), ExpPdu(), ExpPdd(), pX(), pY(),
pYcum(), pYfinal(), ExpX(), VarX(), ExpY(), VarY()
Dim optType, SX, SY, sigX, sigY, T, r, n, vYt, pXh, Row, Price, Prob ', SXY,kx, ky, q
Dim dt, uu, ud, du, dd, emrdt, dx, dy, uX, uY, g, h, i, j, k, l, xbarX, xbarY, etahatX,
etahatY, ut, dmod, vtemp, vtemp2, rho


'**********************************************************
' Read in user inputs
'**********************************************************
Worksheets("Inputs").Activate


Range("A20:XFD1048576").ClearContents
Range("A20:XFD1048576").Borders(xlDiagonalDown).LineStyle = xlNone
Range("A20:XFD1048576").Borders(xlDiagonalUp).LineStyle = xlNone
```

```
Range("A20:XFD1048576").Borders(xlEdgeLeft).LineStyle = xlNone
Range("A20:XFD1048576").Borders(xlEdgeTop).LineStyle = xlNone
Range("A20:XFD1048576").Borders(xlEdgeBottom).LineStyle = xlNone
Range("A20:XFD1048576").Borders(xlEdgeRight).LineStyle = xlNone
Range("A20:XFD1048576").Borders(xlInsideVertical).LineStyle = xlNone
Range("A20:XFD1048576").Borders(xlInsideHorizontal).LineStyle = xlNone

optType = Range("B4").Value
SX = Application.Ln(Range("C5").Value)
SY = Application.Ln(Range("D5").Value)
r = Range("B7").Value
'SXY = Cells(5, 4).Value
'kx = Range("b6").Value
'ky = Range("c6").Value
sigX = Range("C9").Value
sigY = Range("D9").Value
T = Range("B10").Value
n = Range("B11").Value
'q = Cells(8, 2).Value
rho = Range("B13").Value
xbarX = Application.Ln(Range("C14").Value)
xbarY = Application.Ln(Range("D14").Value)
```

```
etahatX = Range("C15").Value

etahatY = Range("D15").Value


'**********************************************************

' Computations

'**********************************************************

'Redimension the arrays based on number of steps specified

'Arrays I have changed the dimension of -wfl

ReDim Opt(1 To n * 2, 1 To n * 2)

ReDim StkX(1 To n * 2 + 1)

ReDim StkY(1 To n * 2 + 1)

ReDim StkXpa(1 To n * 2 + 1)

ReDim StkYpa(1 To n * 2 + 1)

ReDim vX(1 To n * 2 + 1)

ReDim vY(1 To n * 2 + 1, 1 To n * 2 + 1)



ReDim puu(0 To n, n * 2 + 1, n * 2 + 1)

ReDim pud(0 To n, n * 2 + 1, n * 2 + 1)

ReDim pdu(0 To n, n * 2 + 1, n * 2 + 1)

ReDim pdd(0 To n, n * 2 + 1, n * 2 + 1)
```

```
ReDim CostX(0 To n)

ReDim ChangeX(0 To n)

ReDim CuuXtemp(0 To n, n * 2 + 1, n * 2 + 1)

ReDim CudXtemp(0 To n, n * 2 + 1, n * 2 + 1)

ReDim CduXtemp(0 To n, n * 2 + 1, n * 2 + 1)

ReDim CddXtemp(0 To n, n * 2 + 1, n * 2 + 1)

ReDim Count(0 To n, n * 2 + 1, n * 2 + 1)

ReDim Totalprob(0 To n, n * 2 + 1, n * 2 + 1)

ReDim Timeprob(0 To n)

ReDim Movmnt(0 To n, n * 2 + 1, n * 2 + 1)

ReDim Expectedmovmnt(0 To n)

ReDim ExpPuutemp(0 To n, n * 2 + 1, n * 2 + 1)

ReDim ExpPudtemp(0 To n, n * 2 + 1, n * 2 + 1)

ReDim ExpPdutemp(0 To n, n * 2 + 1, n * 2 + 1)

ReDim ExpPddtemp(0 To n, n * 2 + 1, n * 2 + 1)

ReDim ExpPuu(0 To n)

ReDim ExpPud(0 To n)

ReDim ExpPdu(0 To n)

ReDim ExpPdd(0 To n)

ReDim pX(0 To n, n * 2 + 1 + 1)

ReDim pY(0 To n, n * 2 + 1 + 1, n * 2 + 1 + 1)

ReDim pYcum(0 To n, n * 2 + 1 + 1, n * 2 + 1 + 1)
```

```
ReDim pYfinal(0 To n, n * 2 + 1 + 1)

ReDim ExpX(n)

ReDim VarX(n)

ReDim ExpY(n)

ReDim VarY(n)


dt = T / n 'Step size in years

dx = sigX * Sqr(dt)

dy = sigY * Sqr(dt)

'uX = dx

'uY = dy

'uu = dx + dy    'Up Up movement

'ud = dx - dy    'Up Down movement

'du = -dx + dy   'Down Up movement

'dd = -dx - dy   'Down Down movement

'd = 1 / u 'Down movement multiplier

'emrdt = Exp(-r * dt) 'Discount factor per step


'p is risk neutral probabililty of up movement


StkX(1) = (SX) - n * dx 'Lowest value of X

StkY(1) = (SY) - n * dy 'Lowest value of Y
```

```
CostX(0) = SX

'Generate stock price tree for Stocks X & Y

For j = 1 To n * 2
    StkX(j + 1) = StkX(j) + dx  'Populate every value of StkX from the most negative to
the most positive, based on the number of steps and the size of a step, which is constant
    StkY(j + 1) = StkY(j) + dy

'Filling the grid of possible values in the first time period
Cells(22, 1).Value = "puu"
Cells(22, 1).HorizontalAlignment = xlRight
    Cells(20 + 2, 2).Value = Exp(StkX(1))
    Cells(20 + j + 2, 2).Value = Exp(StkX(j + 1))
     Cells(20 + 2, 2).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-
""??_);_(@_)"
     Cells(20 + j + 2, 2).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-
""??_);_(@_)"
    Cells(20 + 2 * n + 3, 3).Value = Exp(StkY(1))
    Cells(20 + 2 * n + 3, 3 + j).Value = Exp(StkY(j + 1))
     Cells(20 + 2 * n + 3, 3).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-
""??_);_(@_)"
```

```
    Cells(20 + 2 * n + 3, 3 + j).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($*
""-""??_);_(@_)"

Cells(22 + 3 * n, 1).Value = "pud"
Cells(22 + 3 * n, 1).HorizontalAlignment = xlRight
   Cells(20 + (3 * n) + 2, 2).Value = Exp(StkX(1))
   Cells(20 + (3 * n) + j + 2, 2).Value = Exp(StkX(j + 1))
    Cells(20 + (3 * n) + 2, 2).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-
""??_);_(@_)"
    Cells(20 + (3 * n) + j + 2, 2).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($*
""-""??_);_(@_)"
   Cells(20 + (3 * n) + 2 * n + 3, 3).Value = Exp(StkY(1))
   Cells(20 + (3 * n) + 2 * n + 3, 3 + j).Value = Exp(StkY(j + 1))
    Cells(20 + (3 * n) + 2 * n + 3, 3).NumberFormat = "_($* #,##0.00_);_($*
(#,##0.00);_($* ""-""??_);_(@_)"
    Cells(20 + (3 * n) + 2 * n + 3, 3 + j).NumberFormat = "_($* #,##0.00_);_($*
(#,##0.00);_($* ""-""??_);_(@_)"

Cells(22 + 6 * n, 1).Value = "pdu"
Cells(22 + 6 * n, 1).HorizontalAlignment = xlRight
   Cells(20 + (6 * n) + 2, 2).Value = Exp(StkX(1))
   Cells(20 + (6 * n) + j + 2, 2).Value = Exp(StkX(j + 1))
```

```
     Cells(20 + (6 * n) + 2, 2).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-
""??_);_(@_)"
     Cells(20 + (6 * n) + j + 2, 2).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($*
""-""??_);_(@_)"
    Cells(20 + (6 * n) + 2 * n + 3, 3).Value = Exp(StkY(1))
    Cells(20 + (6 * n) + 2 * n + 3, 3 + j).Value = Exp(StkY(j + 1))
     Cells(20 + (6 * n) + 2 * n + 3, 3).NumberFormat = "_($* #,##0.00_);_($*
(#,##0.00);_($* ""-""??_);_(@_)"
     Cells(20 + (6 * n) + 2 * n + 3, 3 + j).NumberFormat = "_($* #,##0.00_);_($*
(#,##0.00);_($* ""-""??_);_(@_)"

Cells(22 + 9 * n, 1).Value = "pdd"
Cells(22 + 9 * n, 1).HorizontalAlignment = xlRight
    Cells(20 + (9 * n) + 2, 2).Value = Exp(StkX(1))
    Cells(20 + (9 * n) + j + 2, 2).Value = Exp(StkX(j + 1))
     Cells(20 + (9 * n) + 2, 2).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-
""??_);_(@_)"
     Cells(20 + (9 * n) + j + 2, 2).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($*
""-""??_);_(@_)"
    Cells(20 + (9 * n) + 2 * n + 3, 3).Value = Exp(StkY(1))
    Cells(20 + (9 * n) + 2 * n + 3, 3 + j).Value = Exp(StkY(j + 1))
```

```
    Cells(20 + (9 * n) + 2 * n + 3, 3).NumberFormat = "_($* #,##0.00_);_($*
(#,##0.00);_($* ""-""??_);_(@_)"
    Cells(20 + (9 * n) + 2 * n + 3, 3 + j).NumberFormat = "_($* #,##0.00_);_($*
(#,##0.00);_($* ""-""??_);_(@_)"

Next j

'Filling the grids of possible values in subsequent time periods
For l = 1 To n
   For j = 1 To n * 2 + 1
      'This puts the prices up above
      Cells(21 + j, l * (n + 2) * 2 + 2).Value = Exp(StkX(j))
      Cells(21 + 2 * n + 2, l * (n + 2) * 2 + 2 + j).Value = Exp(StkY(j))
       Cells(21 + j, l * (n + 2) * 2 + 2).NumberFormat = "_($* #,##0.00_);_($*
(#,##0.00);_($* ""-""??_);_(@_)"
       Cells(21 + 2 * n + 2, l * (n + 2) * 2 + 2 + j).NumberFormat = "_($* #,##0.00_);_($*
(#,##0.00);_($* ""-""??_);_(@_)"

      Cells(20 + (3 * n) + j + 1, l * (n + 2) * 2 + 2).Value = Exp(StkX(j))
      Cells(20 + (3 * n) + 2 * n + 3, l * (n + 2) * 2 + 2 + j).Value = Exp(StkY(j))
       Cells(20 + (3 * n) + j + 1, l * (n + 2) * 2 + 2).NumberFormat = "_($*
#,##0.00_);_($* (#,##0.00);_($* ""-""??_);_(@_)"
```

```
        Cells(20 + (3 * n) + 2 * n + 3, l * (n + 2) * 2 + 2 + j).NumberFormat = "_($*
#,##0.00_);_($* (#,##0.00);_($* ""-""??_);_(@_)"

        Cells(20 + (6 * n) + j + 1, l * (n + 2) * 2 + 2).Value = Exp(StkX(j))
        Cells(20 + (6 * n) + 2 * n + 3, l * (n + 2) * 2 + 2 + j).Value = Exp(StkY(j))
        Cells(20 + (6 * n) + j + 1, l * (n + 2) * 2 + 2).NumberFormat = "_($*
#,##0.00_);_($* (#,##0.00);_($* ""-""??_);_(@_)"
        Cells(20 + (6 * n) + 2 * n + 3, l * (n + 2) * 2 + 2 + j).NumberFormat = "_($*
#,##0.00_);_($* (#,##0.00);_($* ""-""??_);_(@_)"

        Cells(20 + (9 * n) + j + 1, l * (n + 2) * 2 + 2).Value = Exp(StkX(j))
        Cells(20 + (9 * n) + 2 * n + 3, l * (n + 2) * 2 + 2 + j).Value = Exp(StkY(j))
        Cells(20 + (9 * n) + j + 1, l * (n + 2) * 2 + 2).NumberFormat = "_($*
#,##0.00_);_($* (#,##0.00);_($* ""-""??_);_(@_)"
        Cells(20 + (9 * n) + 2 * n + 3, l * (n + 2) * 2 + 2 + j).NumberFormat = "_($*
#,##0.00_);_($* (#,##0.00);_($* ""-""??_);_(@_)"

        'This puts the prices down below
Cells(22 + 22 * n, 1).Value = "X Price Probabilites"
Cells(22 + 22 * n, 1).HorizontalAlignment = xlRight
        Cells(21 + 22 * n + j, l * 3 - 1) = Exp(StkX(j))
```

```
        Cells(21 + 22 * n + j, l * 3 - 1).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-""??_);_(@_)"
Cells(25 + 25 * n, 1).Value = "Y Price Probabilites"
Cells(25 + 25 * n, 1).HorizontalAlignment = xlRight
       Cells(24 + 25 * n + j, l * 3 - 1) = Exp(StkY(j))
        Cells(24 + 25 * n + j, l * 3 - 1).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-""??_);_(@_)"

   Next
Next

' Generate probilities on the branches of the stock price tree
For l = 0 To n Step 1
       For g = -l To l Step 2
                j = g + n + 1

     'First doing Stock X
              vX(j) = etahatX * (xbarX - StkX(j)) - 0.5 * sigX ^ 2

              pu = Application.Max(0, Application.Min(1, 0.5 + 0.5 * vX(j) * dt / dx))
              pd = Application.Max(0, Application.Min(1, 0.5 - 0.5 * vX(j) * dt / dx))
```

```
If l = 0 Then
pX(l, j + 1) = pu
pX(l, j - 1) = pd
Cells(22 + 15 * n - 1, 1).Value = "pX movements"
Cells(22 + 15 * n - 1, 1).HorizontalAlignment = xlRight

Else

pX(l, j + 1) = pX(l, j + 1) + pX(l - 1, j) * pu
pX(l, j - 1) = pX(l, j - 1) + pX(l - 1, j) * pd

End If

'This puts the X Price Probabilities in the tree above
Cells(22 + 16 * n - j - 1, 2 + l) = pX(l, j + 1)
Cells(22 + 16 * n - j + 1, 2 + l) = pX(l, j - 1)
 Cells(22 + 16 * n - j - 1, 2 + l).Style = "Percent"
 Cells(22 + 16 * n - j - 1, 2 + l).NumberFormat = "0.0%"
 Cells(22 + 16 * n - j + 1, 2 + l).Style = "Percent"
 Cells(22 + 16 * n - j + 1, 2 + l).NumberFormat = "0.0%"

'This puts the X Price Probabilities in the list below
```

```
            Cells(21 + 22 * n + j + 1, l * 3 + 3) = pX(l, j + 1)
            Cells(21 + 22 * n + j - 1, l * 3 + 3) = pX(l, j - 1)
            Cells(21 + 22 * n + j + 1, l * 3 + 3).Style = "Percent"
            Cells(21 + 22 * n + j + 1, l * 3 + 3).NumberFormat = "0.0%"
            Cells(21 + 22 * n + j - 1, l * 3 + 3).Style = "Percent"
            Cells(21 + 22 * n + j - 1, l * 3 + 3).NumberFormat = "0.0%"

    'Now doing Stock Y
          For h = -l To l Step 2
          k = h + n + 1
'         Calculate vX and vY based on the formulas on p 538, col 1
'         Use it to calculate the probability of moves up, down, p 538 column 2
'         Then use that to multiply by the probability of moves u|u, u|d, d|u, and d|d, to give
the puu, pud, pdu and pdd, p 538 column 2

          vY(j, k) = etahatY * (xbarY - StkY(k)) - 0.5 * sigY ^ 2

          puIFu = Application.Max(0, Application.Min(1, (dx * (dy + dt * vY(j, k)) + dt
* (dy * vX(j) + rho * sigX * sigY)) / (2 * dy * (dx + dt * vX(j)))))
            puu(l, j, k) = pu * puIFu
          pdIFu = Application.Max(0, Application.Min(1, (dx * (dy - dt * vY(j, k)) + dt *
(dy * vX(j) - rho * sigX * sigY)) / (2 * dy * (dx + dt * vX(j)))))
```

```
                pud(l, j, k) = pu * pdIFu
            puIFd = Application.Max(0, Application.Min(1, (dx * (dy + dt * vY(j, k)) - dt *
(rho * sigX * sigY + dy * vX(j))) / (2 * dy * (dx - dt * vX(j)))))
                pdu(l, j, k) = pd * puIFd
            pdIFd = Application.Max(0, Application.Min(1, (dx * (dy - dt * vY(j, k)) + dt *
(rho * sigX * sigY - dy * vX(j))) / (2 * dy * (dx - dt * vX(j)))))
                pdd(l, j, k) = pd * pdIFd
'           pdd(l, j, k) = 1 - puu(l, j, k) - pud(l, j, k) - pdu(l, j, k)

'           If j < n * 2 + 1 And k < n * 2 + 1 And j > 1 And k > 1 Then



            Cells(21 + j, 2 + k + l * (n + 2) * 2).Value = puu(l, j, k)
            Cells(21 + (3 * n) + j, 2 + k + l * (n + 2) * 2).Value = pud(l, j, k)
            Cells(21 + (6 * n) + j, 2 + k + l * (n + 2) * 2).Value = pdu(l, j, k)
            Cells(21 + (9 * n) + j, 2 + k + l * (n + 2) * 2).Value = pdd(l, j, k)
             Cells(21 + j, 2 + k + l * (n + 2) * 2).Style = "Percent"
             Cells(21 + j, 2 + k + l * (n + 2) * 2).NumberFormat = "0.0%"
             Cells(21 + (3 * n) + j, 2 + k + l * (n + 2) * 2).Style = "Percent"
             Cells(21 + (3 * n) + j, 2 + k + l * (n + 2) * 2).NumberFormat = "0.0%"
             Cells(21 + (6 * n) + j, 2 + k + l * (n + 2) * 2).Style = "Percent"
             Cells(21 + (6 * n) + j, 2 + k + l * (n + 2) * 2).NumberFormat = "0.0%"
```

```
Cells(21 + (9 * n) + j, 2 + k + l * (n + 2) * 2).Style = "Percent"
Cells(21 + (9 * n) + j, 2 + k + l * (n + 2) * 2).NumberFormat = "0.0%"

If l = 0 Then
pYcum(l, j + 1, k + 1) = puu(l, j, k)
pYcum(l, j - 1, k + 1) = pdu(l, j, k)
pYcum(l, j + 1, k - 1) = pud(l, j, k)
pYcum(l, j - 1, k - 1) = pdd(l, j, k)
pYfinal(l, k + 1) = puu(l, j, k) + pdu(l, j, k) + pYfinal(l, k + 1)
pYfinal(l, k - 1) = pud(l, j, k) + pdd(l, j, k) + pYfinal(l, k - 1)

' don't know what this refers to.  Delete?   Cells(22 + 11 * n - k - 1, 2 + l) =
pYfinal(l, k + 1)
' don't know what thsi refers to. Delete?   Cells(22 + 11 * n - k + 1, 2 + l) =
pYfinal(l, k - 1)
Cells(21 + 19 * n - 1, 1).Value = "pY movements"
Cells(21 + 19 * n - 1, 1).HorizontalAlignment = xlRight

'Puts the Y Price Probabilities above at Time=0
Cells(21 + 20 * n - k - 1, 2 + l) = pYfinal(l, k + 1)
Cells(21 + 20 * n - k + 1, 2 + l) = pYfinal(l, k - 1)
Cells(21 + 20 * n - k - 1, 2 + l).Style = "Percent"
```

```
 Cells(21 + 20 * n - k - 1, 2 + l).NumberFormat = "0.0%"
 Cells(21 + 20 * n - k + 1, 2 + l).Style = "Percent"
 Cells(21 + 20 * n - k + 1, 2 + l).NumberFormat = "0.0%"

'Puts the Y Price Probabitlies below at Time=0
Cells(24 + 25 * n + k + 1, l * 3 + 3) = pYfinal(l, k + 1)
Cells(24 + 25 * n + k - 1, l * 3 + 3) = pYfinal(l, k - 1)
 Cells(24 + 25 * n + k + 1, l * 3 + 3).Style = "Percent"
 Cells(24 + 25 * n + k + 1, l * 3 + 3).NumberFormat = "0.0%"
 Cells(24 + 25 * n + k - 1, l * 3 + 3).Style = "Percent"
 Cells(24 + 25 * n + k - 1, l * 3 + 3).NumberFormat = "0.0%"

Else

pYcum(l, j + 1, k + 1) = pYcum(l - 1, j, k) * puu(l, j, k) + pYcum(l, j + 1, k + 1)
pYcum(l, j - 1, k + 1) = pYcum(l - 1, j, k) * pdu(l, j, k) + pYcum(l, j - 1, k + 1)
pYcum(l, j + 1, k - 1) = pYcum(l - 1, j, k) * pud(l, j, k) + pYcum(l, j + 1, k - 1)
pYcum(l, j - 1, k - 1) = pYcum(l - 1, j, k) * pdd(l, j, k) + pYcum(l, j - 1, k - 1)
```

```
        End If

        Next

      Next

    Next

      For l = 1 To n Step 1

        For g = -l To l Step 2

        j = g + n + 1

          For h = -l To l Step 2

          k = h + n + 1



          pYfinal(l, k + 1) = pYcum(l - 1, j, k) * puu(l, j, k) + pYcum(l - 1, j, k) *
pdu(l, j, k) + pYfinal(l, k + 1)

          pYfinal(l, k - 1) = pYcum(l - 1, j, k) * pud(l, j, k) + pYcum(l - 1, j, k) *
pdd(l, j, k) + pYfinal(l, k - 1)

        'Puts the Y Price Probabilities above

        Cells(21 + 20 * n - k - 1, 2 + l) = pYfinal(l, k + 1)

        Cells(21 + 20 * n - k + 1, 2 + l) = pYfinal(l, k - 1)
```

```
            Cells(21 + 20 * n - k - 1, 2 + l).Style = "Percent"

            Cells(21 + 20 * n - k - 1, 2 + l).NumberFormat = "0.0%"

            Cells(21 + 20 * n - k + 1, 2 + l).Style = "Percent"

            Cells(21 + 20 * n - k + 1, 2 + l).NumberFormat = "0.0%"


            'Puts the Y Price Probabilities below


            Cells(24 + 25 * n + k + 1, l * 3 + 3) = pYfinal(l, k + 1)

            Cells(24 + 25 * n + k - 1, l * 3 + 3) = pYfinal(l, k - 1)

            Cells(24 + 25 * n + k + 1, l * 3 + 3).Style = "Percent"

            Cells(24 + 25 * n + k + 1, l * 3 + 3).NumberFormat = "0.0%"

            Cells(24 + 25 * n + k - 1, l * 3 + 3).Style = "Percent"

            Cells(24 + 25 * n + k - 1, l * 3 + 3).NumberFormat = "0.0%"


    'just for reference         Cells(105 - k - 1 + j * 8, 2 + l) = pYfinal(l, k + 1)

        'just for reference      Cells(105 - k + 1 + j * 8, 2 + l) = pYfinal(l, k - 1)


            Next

      Next

Next


'Calculating the Expected Values, Variance, and Std. Dev. by Time
```

```
Cells(23 + 24 * n, 1).Value = "X Expected Value"
Cells(24 + 24 * n, 1).Value = "X Variance"
Cells(25 + 24 * n, 1).Value = "X Std Dev"
 Cells(23 + 24 * n, 1).HorizontalAlignment = xlRight
 Cells(24 + 24 * n, 1).HorizontalAlignment = xlRight
 Cells(25 + 24 * n, 1).HorizontalAlignment = xlRight

Cells(26 + 27 * n, 1).Value = "Y Expected Value"
Cells(27 + 27 * n, 1).Value = "Y Variance"
Cells(28 + 27 * n, 1).Value = "Y Std Dev"
 Cells(26 + 27 * n, 1).HorizontalAlignment = xlRight
 Cells(27 + 27 * n, 1).HorizontalAlignment = xlRight
 Cells(28 + 27 * n, 1).HorizontalAlignment = xlRight



'Expected Value in Each Time Period
For l = 1 To n Step 1
    'For X
    Cells(23 + 24 * n, l * 3).Select
    For Row = -n * 2 - 1 To -1 Step 1
     Price = Application.Ln(ActiveCell.Offset(Row, -1).Value)
```

```
     Prob = ActiveCell.Offset(Row, 0).Value
     ExpX(l) = Price * Prob + ExpX(l)
    Next
    Cells(23 + 24 * n, l * 3) = Exp(ExpX(l))
     Cells(23 + 24 * n, l * 3).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-
""??_);_(@_)"
'    Selection.FormulaArray = "=SUM(R[-7]C[-1]:R[-1]C[-1]*R[-7]C:R[-1]C)"

    'For Y
    Cells(26 + 27 * n, l * 3).Select
    For Row = -n * 2 - 1 To -1 Step 1
     Price = Application.Ln(ActiveCell.Offset(Row, -1).Value)
     Prob = ActiveCell.Offset(Row, 0).Value
     ExpY(l) = Price * Prob + ExpY(l)
    Next
    Cells(26 + 27 * n, l * 3) = Exp(ExpY(l))
     Cells(26 + 27 * n, l * 3).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-
""??_);_(@_)"
'    Selection.FormulaArray = "=SUM(R[-7]C[-1]:R[-1]C[-1]*R[-7]C:R[-1]C)"
Next

'Variance in Each Time Period
```

```
For l = 1 To n Step 1
    'Fox X
    Cells(24 + 24 * n, l * 3).Select
    For Row = -n * 2 - 2 To -2 Step 1
     Price = Application.Ln(ActiveCell.Offset(Row, -1).Value)
     Prob = ActiveCell.Offset(Row, 0).Value
     VarX(l) = ((Price - ExpX(l)) * Prob) ^ 2 + VarX(l)
    Next
    Cells(24 + 24 * n, l * 3) = VarX(l)
     Cells(24 + 24 * n, l * 3).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-""??_);_(@_)"
'    Selection.FormulaArray = "=SUM((R[-8]C[-1]:R[-2]C[-1]-R[-1]C)^2*R[-8]C:R[-2]C)"

    'For Y
    Cells(27 + 27 * n, l * 3).Select
    For Row = -n * 2 - 2 To -2 Step 1
     Price = Application.Ln(ActiveCell.Offset(Row, -1).Value)
     Prob = ActiveCell.Offset(Row, 0).Value
     VarY(l) = ((Price - ExpY(l)) * Prob) ^ 2 + VarY(l)
    Next
    Cells(27 + 27 * n, l * 3) = VarY(l)
```

```
    Cells(27 + 27 * n, l * 3).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-
""??_);_(@_)"
'   Selection.FormulaArray = "=SUM((R[-8]C[-1]:R[-2]C[-1]-R[-1]C)^2*R[-8]C:R[-
2]C)"
Next
'Std Dev in Each Time Period
For l = 1 To n Step 1
   Cells(25 + 24 * n, l * 3).Select
   ActiveCell.FormulaR1C1 = "=SQRT(R[-1]C)"
    Cells(25 + 24 * n, l * 3).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-
""??_);_(@_)"

   Cells(28 + 27 * n, l * 3).Select
   ActiveCell.FormulaR1C1 = "=SQRT(R[-1]C)"
    Cells(28 + 27 * n, l * 3).NumberFormat = "_($* #,##0.00_);_($* (#,##0.00);_($* ""-
""??_);_(@_)"

Next
Cells(1, 1).Select


End Sub
```